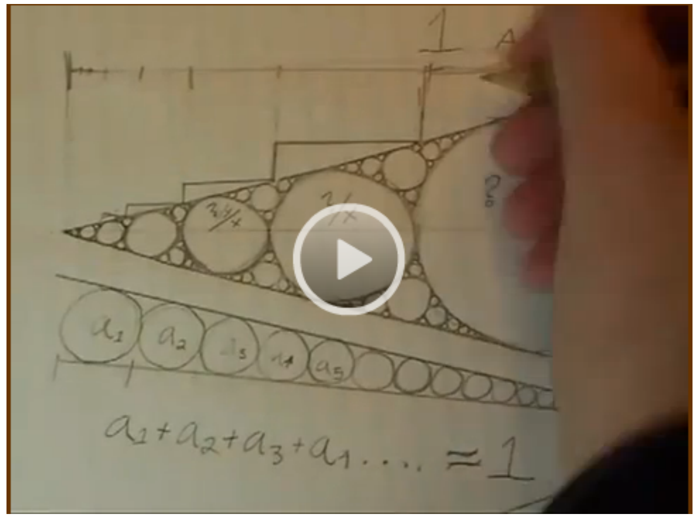
Geometric Series

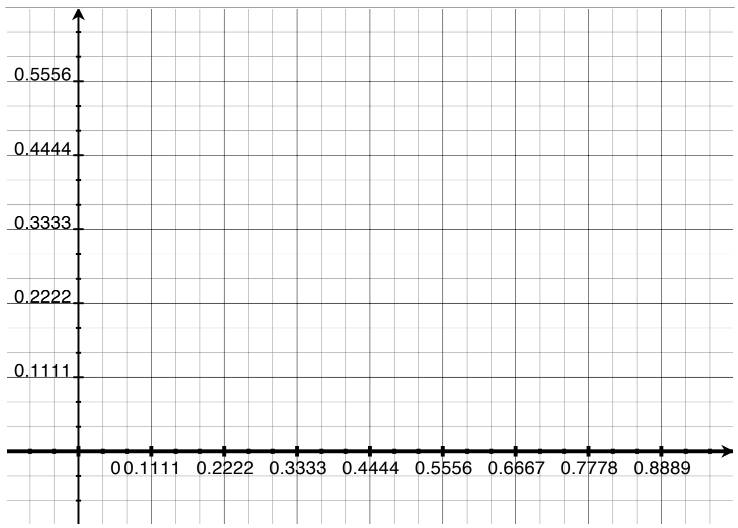
Adapted to complement
ViHart's *Infinity Elephants* video.

1. Consider the sequence $a_n = \frac{1}{3}^n$.



   (a) Write down a few of the terms in the sequence a_n.

   We will follow Vi Hart's lead and record the sequence of partial sums by moving horizontally across the page.



   (b) Draw a box (or an elephant) that is $\frac{1}{3}$ units high & $\frac{1}{3}$ units wide with the lower left "corner" on the origin.

   The height will correspond with a_1 and the total width across the page corresponds with s_1.

   (c) Draw the second box that has height a_2 immediately to the right of the box (or elephant) you drew in (b) (that is with the lower left "corner" at the point $(\frac{1}{3},0)$).

   (d) Identify the length that corresponds with s_2.

   (e) Draw a third box (or elephant) with height and width a_3 immediately to the right of the box (or elephant) you drew in (c).

   (f) Identify the length that corresponds with s_3.

   (g) It gets difficult to repeat the procedure for s_4, s_5 and so on, but can you identify $\lim_{n\to\infty} s_n$ is on the graph above?

   (h) Just as Vi Hart drew a line that skimmed across the tops of the circles, draw a line that skims the top right hand side of the boxes. Find the equation of the line you just drew.

   (i) Find $\lim_{n\to\infty} s_n$

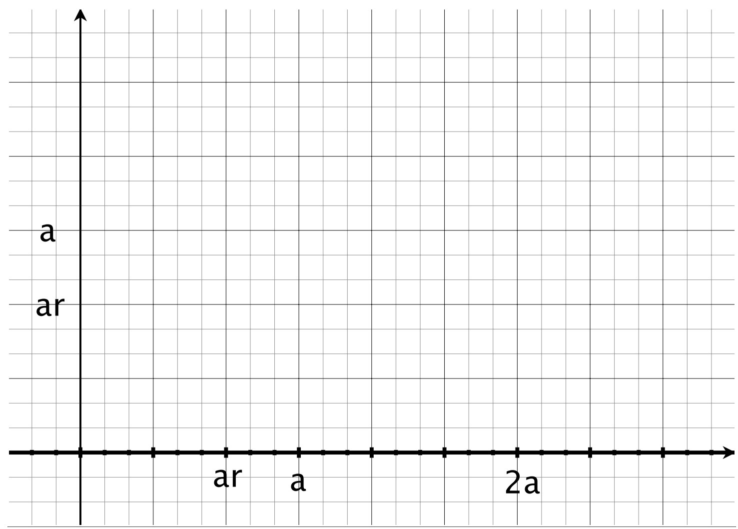
The above is an example of a geometric series. In general, a *geometric series* is an infinite series of the form:

$$a + ar + ar^2 + ar^3 + ar^4 + ar^5 + ... + ar^{n-1} + ... = \sum_{n=1}^{\infty} ar^{n-1}$$

Or, using the definition of a series, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} (a + ar + ar^2 ... + ar^n)$.

In the above example, both a and r were equal to $\frac{1}{3}$. Lets consider the general case...

2. Fix a and r as positive numbers with $r < 1$. We will mimic the steps done on the front side of this worksheet to find what the geometric series converges to in general. Consider the sequence $a_n = ar^{n-1}$.



(a) Write down a few of the terms in the sequence a_n.

(b) Draw a box (or an elephant) that is a_1 units high & a_1 units wide with the lower left "corner" on the origin.

The height will correspond with a_1 and the total width across the page corresponds with s_1.

(c) Draw the second box that has height a_2 immediately to the right of the box (or elephant) you drew in (b) (that is with the lower left "corner" at the point (a,0)). Notice to see the picture better I've labeled what ar is (effectively making r equal $\frac{2}{3}$, but by keeping the r's in the picture, you'll be able to keep the general result).

(d) Identify the length that corresponds with s_2.

(e) Draw a third box (or elephant) immediately to the right of the box (or elephant) you drew in (c).

(f) Identify the length that corresponds with s_3.

(g) It gets difficult to repeat the procedure for s_4, s_5 and so on, but can you identify $\lim_{n\to\infty} s_n$ is on the graph above?

(h) Just as Vi Hart drew a line that skimmed across the tops of the circles, draw a line that skims the top right hand side of the boxes. Find the equation of the line you just drew.

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