Key

Exam 2 **TMath 126** Spring 2025

As a reminder, you are welcome to use a non-internet accessing calculator (which includes Desmos Test Mode) and a one-sided 8.5" by 11" sheet of notes. Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

1. Let $\beta(x,y) = \dfrac{x-2}{x^2y-4y}$. Find the following, if possible.

   (a) [1] $\beta(11,2)$

   $$= \frac{11-2}{11^2\cdot 2-4(2)} = \frac{9}{242-8} = \frac{9}{234} = \frac{3}{78} = \frac{1}{26} \approx .03846$$

   plug in +1

   (b) [2] (§14.2 #46) a point *not* in the domain of β.

   There are lots of points eg $(5,0)$ b/c zero in denominator
   $(2,1)$ b/c "0/0"

   Know domain +.5 point +1 notation +.5

   (c) [2] (LimitActivity#3) $\lim_{(x,y)\to(2,3)} \beta(x,y)$

   $$\underbrace{\frac{2-2}{4(3)-4(3)} = \frac{\text{"}0\text{"}}{0}}_{+.5} \Rightarrow \text{numerically or algebraically}$$

   got it +.5

x	y	$\beta(x,y)$
1.999	3	.08335
2.0001	3	.08333

   so $\approx .083$

   table +1

   $$\beta(x,y) = \frac{(x-2)}{y(x^2-4)} = \frac{\cancel{x-2}}{y(x+2)\cancel{(x-2)}}$$

   $$\Rightarrow \lim_{(x,y)\to(2,3)} \frac{1}{y(x+2)} = \frac{1}{3(2+2)} = \frac{1}{12} = .08\overline{33}$$

   algebra +1

   (d) [2] (Quiz5#2) $\beta_y(x,y)$

   ie $\frac{\partial}{\partial y}\beta(x,y)$

   +.5 treat x as a constant

   $$\beta(x,y) = \left(\frac{x-2}{x^2-4}\right)\frac{1}{y} \Rightarrow \beta_y(x,y) = \overbrace{\left(\frac{x-2}{x^2-4}\right)}^{+.5} \cdot \overbrace{-1y^{-2}}^{+1}$$

   (e) [1] (WebHW14.4#1) $\beta_y(1,4)$

   $\beta_y(1,4)$ plug in $(1,4)$ to answer from (d) +1

   $$= \left(\frac{1-2}{1-4}\right)\frac{-1}{4^2} = \frac{-1}{-3}\cdot\frac{-1}{16} = \frac{-1}{48}$$

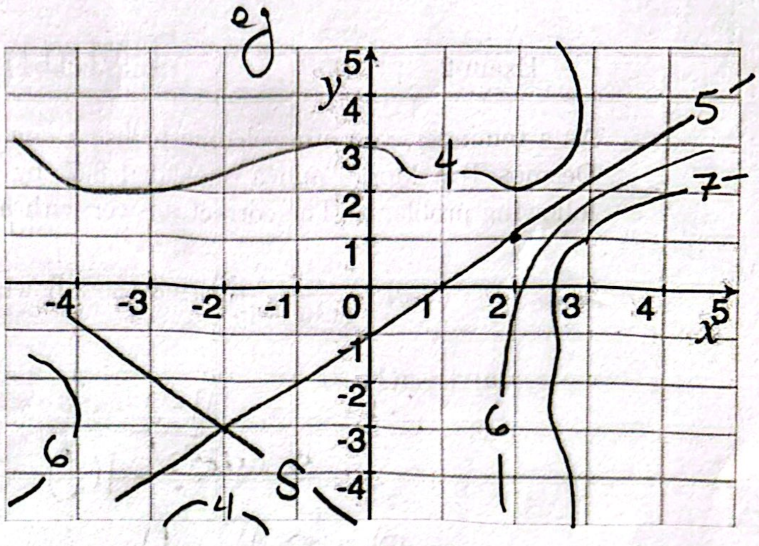
8

2. [5] Draw the contours of a function f of two variables (x, y), that satisfies *all* of the following:

(+1)

Note there are lots of correct answers

- (+1) (a) $f(2,1) = 5$
- (+1) (b) $f_x(2,1) \approx 2$
- (+1) (c) $f_y(2,1) \approx -1$
- (+1) (d) $(-2,-3)$ is a saddle point



3. Let $\vec{u} = \frac{3}{\sqrt{10}}\vec{i} - \frac{1}{\sqrt{10}}\vec{j}$ and let α be a differentiable function with

- $\alpha(-3,2) = 4$
- $\alpha_x(-3,2) = -1$
- $\alpha_y(-3,2) = 3$

(a) [3] (TangentActivity #1) Find a linearization for α when $x = -3$ and $y = 2$.

Looking for $z - z_1 = M_x(x - x_1) + M_y(y - y_1)$ form (+.5)

(+.5) (+.5) (+.5) (+.5) (+.5)

$z - 4 = -1(x - -3) + 3(y - 2)$

(b) [2] (WebHW§14.4 #3) Use your linearization to approximate $\alpha(-3.2, 1.9)$.

ie plug in $x = -3.2$ & $y = 1.9$

$? - 4 = -(-3.2 + 3) + 3(1.9 - 2)$

$\Rightarrow ? = 3.9$

(c) [2] (PracticeExam#4) Find $D_{\vec{u}}\alpha(-3,2)$.

$D_{\vec{u}}\alpha(-3,2) = \nabla\alpha\big|_{(-3,2)} \cdot \vec{u}$ (+.5)

OR

(note $\|\vec{u}\| = \sqrt{\left(\frac{3}{\sqrt{10}}\right)^2 + \left(\frac{-1}{\sqrt{10}}\right)^2} = 1$)

$D_{\vec{u}}\alpha(-3,2) = \langle f_x\big|_{(-3,2)}, f_y\big|_{(-3,2)} \rangle \cdot \langle \frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}} \rangle$ (+.5)

$= \langle -1, 3 \rangle \cdot \langle \frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}} \rangle$ (+.5)

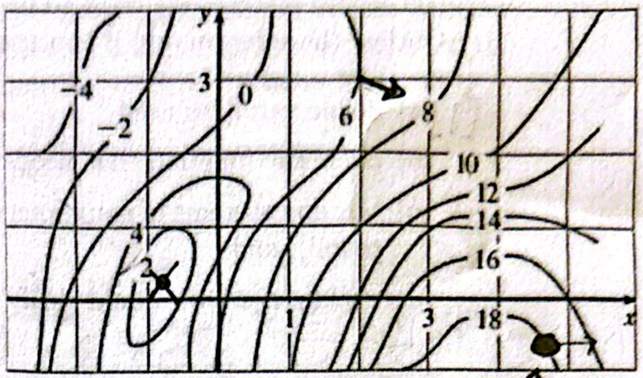
$= \frac{-3}{\sqrt{10}} - \frac{3}{\sqrt{10}} = \frac{-6}{\sqrt{10}}$ dotprod (+.5)

≈ -1.897

12

Sketch The direction of the gradient vector

4. (§10.1 #58) Let g have the contour lines shown on the right.



(a) [1] (OptimizeActivity#1) Identify a critical point

X mark about $(-.8, .2), 1.7)$

(b) [2] (§14.6 #26) Sketch the gradient vector at $(2, 3)$ (+1) steepest ascent (+.5) direction (+.5)

(c) [2] (PracticeExam#2) Identify a point (a, b) where $g_x(a, b) < 0$.

(+.5) need elevation ↓

(+.5) as move to right

got one (+1)

(a, b)

5. A particular hill's elevation is well modeled by $z = 2000 - 0.005x^2 - .01y^2$ where x, y, and z are measured in meters. The positive x-axis points east and the positive y-axis points north.

(a) [1] What is your elevation at $x = 160$ and $y = 160$?

$2000 - .005(160)^2 - .01(160)^2 = 1616$ meters

(b) [2] (§14.3#2) What is the meaning of $\frac{\partial z}{\partial y}$ in terms of walking and elevation?

(+.5) rates of change

$\partial z / \partial y$ records the elevation change as we walk north (+1)

(+.5) (as a function of where we start)

(c) [3] If you walk east, from the point $x = 160$ and $y = 160$, do you ascend or descend? Provide justification.

start (+.5)

(+.5) i.e. what is sign of $\left.\frac{\partial z}{\partial x}\right|_{(160,160)}$

Could also graph some contour lines

$\frac{\partial z}{\partial x} = -.005(2)x$ (+1) $\Rightarrow \left.\frac{\partial z}{\partial x}\right|_{(160,160)} = -1.6$ (use point +.5) $\Rightarrow$ descend (+.5)

(d) [3] (WebHW14.3 #6) In which direction should you go to increase your elevation the quickest with one step?

direction (+.5)

(+.5) ie what is $\nabla f(160, 160) = \langle f_x|_{(160,160)}, f_y|_{(160,160)} \rangle$

$f_x(x, y) = -.005 \cdot 2x = -.01x$

(+1) $f_y(x, y) = -.01 \cdot 2y = -.02y$ @ (160,160)

$= \langle -1.6, -3.2 \rangle$ (+.5) (+.5)

south by southwest (+.5)

When starting from $x = 160$ and $y = 160$

14

$\frac{20 + 20}{40}$ Median 70%

6. [6] (Suggested§14.8 #55) A grain silo is to be built by attaching a hemispherical roof and a flat floor onto a circular cylinder. The material (surface area) is restricted to 800 square meters. We want to find the dimensions of the silo that maximize volume. Outline the solution and if you use chapter 14 techniques, make sure to:

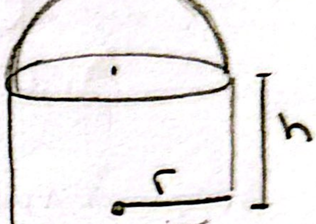
[1] (a) define variables used

(b) circle the function that needs to be optimized

(c) box any systems of equations that need to be solved (but you do not need to solve them!), and

(d) explain how you would verify you have a maximum.

(a)



picture (+.5)
variables (+.5)
start (+.5)

	Surface Area	Volume
sphere	$4\pi r^2$	$\frac{4}{3}\pi r^3$
cylinder	$2\pi r \cdot h$	$\pi r^2 \cdot h$
	+.5	+.5

b) maximize volume = vol of cylinder + vol of hemisphere (+.5)

$M(r,h) = \pi r^2 \cdot h + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$ (+.5)

Know surface area = 800 m²

$\Rightarrow 800\,m^2$ = floor + walls + ceiling $= C(r,h)$ (+1)

$= \pi r^2 + 2\pi r \cdot h + \left(\frac{1}{2}\right)(4\pi r^2)$ (*)

Could use (*) to solve for h and sub into max vol eq. (+.5)
Then use MATH 124 techniques

(*) $\Rightarrow h = \frac{1}{2\pi r}(800 - \pi r^2 - 2\pi r^2)$ (+.5)

$\Rightarrow M(r) = \pi r^2\left(\frac{1}{2\pi r}(800 - \pi r^2 - 2\pi r^2)\right) + \frac{2}{3}\pi r^3$ (+1)

(don't see a max? (+.5)

OR §14.8 Lagrange Mult

$\begin{cases} \nabla M = \lambda \nabla C \\ C(r,h) = 800 \end{cases}$ (+.5) (+.5)

$\Downarrow$

$$\boxed{\begin{aligned} 2\pi r h + 2\pi r^2 &= (2\pi r + 2\pi h + 4\pi r)\lambda \\ \pi r^2 + 0 &= (2\pi r)\lambda \\ 800 &= \pi r^2 + 2\pi r h + 2\pi r^2 \end{aligned}}$$ (+.5)

(d) Once I have the points I'll plug them + some close points into M(r,h) to me...