EXAM 2

TMath 124

Practice

Key

Note: This is a practice midterm and is intended only for study purposes. The actual exam will contain different questions and perhaps a different layout.

1. [] TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F. Let f and g be functions.

T F $\frac{d}{dx}b^c = cb^{c-1}$ for a fixed b and c $\quad b^c$ is a constant so $\frac{d}{dx}(b^c)=0$

T F $(x+y)^2 = x^2+y^2$ $\quad (x+y)^2=(x+y)(x+y)=x^2+xy+xy+y^2$

T F $\frac{d}{dx}2^x = x2^{x-1}$ $\quad \frac{d}{dx}(2^x) = 2^x(\ln 2)$

Show your work for the following problems. The correct answer with no supporting work will receive NO credit (this includes multiple choice questions).

2. [] Sketch the graph of an example function f that satisfies the following conditions:

(a) f is not differentiable when $x=-3$ ✓

(b) f is continuous when $x=-3$ ✓

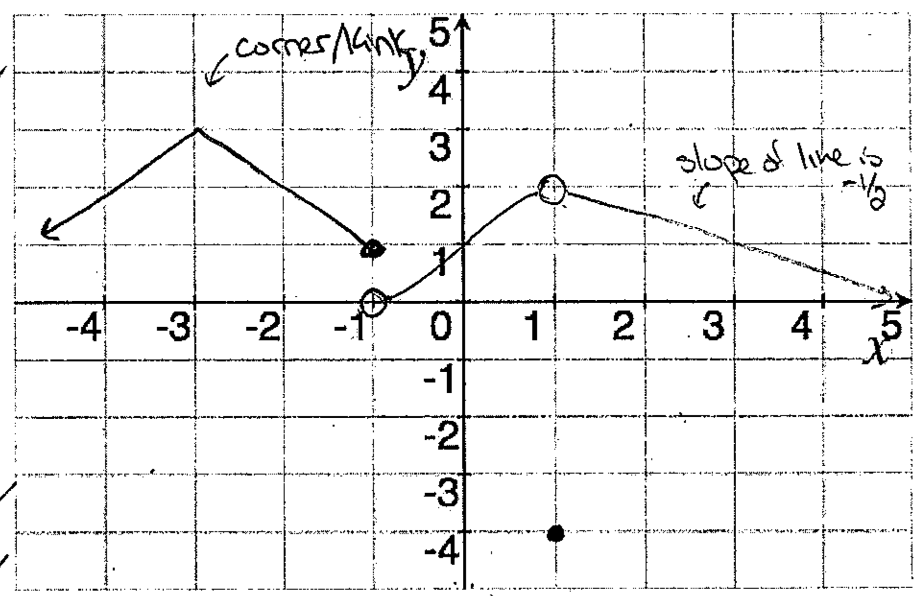
(c) $f(1)=-4$ ✓

(d) $\lim_{x\to 1} f(x) = 2$ ✓

(e) $f'(3) = -\frac{1}{2}$ ✓

(f) $\lim_{x\to -1^+} f(x) = 0$ ✓

(g) $\lim_{x\to -1^-} f(x) = 1$ ✓



Find a formula for the above graph.

$$f(x)=\begin{cases} x+6 & \text{if } x\le -3 \\ -x & \text{if } -3\le x\le -1 \\ x+1 & \text{if } -1<x<1 \\ -4 & \text{if } x=1 \\ -\frac{1}{2}x+2.5 & \text{if } 1<x \end{cases}$$

$\sin^2 x + \cos^2 x = 1$

$\Rightarrow \quad \cos^2 x - 1 = -\sin^2$

3. Find the following:

$$\lim_{x\to 0} \frac{3\sin(4x)}{2\sin(3x)} \cdot \frac{2x}{2x} = \lim_{x\to 0}\left(\frac{\sin 4x}{4x}\right)\left(\frac{3\cdot 2x}{\sin 3x}\right)$$

$$= \lim_{x\to 0} \frac{\sin 4x}{4x} \cdot \lim_{x\to 0}\left(\frac{3x}{\sin 3x}\right)\cdot 2$$

$$= \lim_{x\to 0} \frac{1}{\frac{\sin 3x}{3x}} \lim_{x\to 0} 2 = 2$$

$$\lim_{x\to 0} \frac{\cos x - 1}{\sin x} \frac{(\cos x + 1)}{(\cos x + 1)}$$

$$= \lim_{x\to 0} \frac{\cos^2 x - 1}{\sin(x)(\cos(x)+1)}$$

$$= \lim_{x\to 0} \frac{-\sin^{\cancel{2}} x}{\cancel{\sin(x)}(\cos(x)+1)} = \lim_{x\to 0} \frac{-\sin x}{\cos(x)+1}$$

$$= \frac{-0}{2} = 0$$

4. Suppose that $f(2) = -3$, $g(2) = 4$, $f'(2) = -2$, and $g'(2) = 7$. Find $h'(2)$ where h is:

$h(x) = 5f(x) - 4g(x)$

$$h'(x) = [5f(x) - 4g(x)]'$$
$$= [5f(x)]' - [4g(x)]'$$
$$= 5f'(x) - 4g'(x)$$
$$\Rightarrow h'(2) = 5f'(2) - 4g'(2)$$
$$= 5(-2) - 4(7) = -10 - 28 = -38$$

$h(x) = f(x)g(x)$ product rule

$$h'(x) = [f(x)g(x)]' = f(x)g'(x) + f'(x)g(x)$$
$$\Rightarrow h'(2) = f(2)g'(2) + f'(2)g(2)$$
$$= (-3)(7) + (-2)(4)$$
$$= -21 - 8 = -29$$

$h(x) = \frac{f(x)}{g(x)}$ quotient rule

$$h'(x) = \left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\Rightarrow h'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{[g(2)]^2} = \frac{(4)(-2) - (-3)(7)}{4^2}$$

$$= (-8+21)/16 = 13/16$$

$h(x) = \frac{g(x)}{1+f(x)}$ quotient rule

$$h'(x) = \left[\frac{g(x)}{1+f(x)}\right]' = \frac{[1+f(x)]g'(x) - g(x)[1+f(x)]'}{[1+f(x)]^2}$$

$$\Rightarrow h'(2) = \frac{[1+f(2)]g'(2) - g(2)f'(2)}{[1+f(2)]^2}$$

$$= \frac{[1+(-3)](7) - 4(-2)}{[1+(-3)]^2} = \frac{-14+8}{4} = \frac{-3}{2}$$

5. If $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, and $g'(5) = 6$, find $F'(5)$.

$$F'(x) = f'(g(x))\,g'(x) \text{ by chain rule}$$

$$F'(5) = f'(g(5))g'(5) = f'(-2)\cdot 6 = 4\cdot 6 = 24$$

note: $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x(\sin x)' - \sin(x)(\cos x)'}{(\cos x)^2}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$ ✓

6. Find the $\frac{dy}{dx}$ of the following:

$y = (2x^2 + 7x^2)(3^x - 2^x)$ product rule

$= (9x^2)(3^x - 2^x)$

$\frac{dy}{dx} = (9x^2)\frac{d}{dx}(3^x - 2^x) + \frac{d}{dx}(9x^2)(3^x - 2^x)$

$= 9x^2\left[\frac{d}{dx}(3^x) - \frac{d}{dx}(2^x)\right] + 18x(3^x - 2^x)$

$= 9x^2[3^x(\ln 3) - 2^x(\ln 2)] + 18x(e^x - 2^x)$

$y = \frac{\sin(x) + x^2\cos(x)}{\cos(x)}$

$= \frac{\sin(x)}{\cos(x)} + \frac{x^2\cos(x)}{\cos x}$ note: domain remains

$= \tan(x) + x^2$

$\frac{dy}{dx} = \frac{d}{dx}[\tan(x) + x^2]$

$= \frac{d}{dx}[\tan(x)] + \frac{d}{dx}(x^2)$

$= \sec^2 x + 2x$

$y = \sqrt{\frac{x^2+1}{4x^5 - 3x}}$ $\underbrace{\text{quotient rule}}$ see below ✗

Chain

inside $g(x) = \frac{x^2+1}{4x^5-3x}$ $g'(x) =$

outside $f(u) = u^{1/2}$ $f'(u) = \frac{1}{2}u^{-1/2}$

$f \circ g(x)$ ✓

$\frac{dy}{dx} = f'(g(x))g'(x) = \frac{1}{2}\left(\frac{x^2+1}{4x^5-3x}\right)^{-1/2}\left[\frac{(4x^5-3x)2x - (x^2+1)(20x^4-3)}{[4x^5-3x]^2}\right]$

$g'(x) = \frac{(4x^5-3x)[x^2+1]' - [x^2+1](4x^5-3x)'}{[4x^5-3x]^2}$

$= \frac{(4x^5-3x)\,2x - (x^2+1)(20x^4-3)}{[4x^5-3x]^2}$

$e^y\sin(x) = x + xy$

$\frac{d}{dx}(e^y\sin(x)) = \frac{d}{dx}(x + xy)$

product rule

$e^y\frac{d}{dx}(\sin x) + \frac{d}{dx}(e^y)\sin x = \frac{d}{dx}(x) + \frac{d}{dx}(xy)$

$e^y\cos x + e^y y'\sin x = 1 + \frac{d}{dx}(xy)$ product rule

$e^y\cos x + y'e^y\sin x = 1 + x\frac{d}{dx}(y) + \frac{d}{dx}(x)y$

$e^y\cos x + y'e^y\sin x = 1 + xy' + y$

$-e^y\cos x \quad -xy' \quad -xy' \quad -e^y\cos x$

$y' = \frac{1 + y - e^y\cos x}{e^y\sin x - x}$

$y = \sin(e^{\sin(x^2)})$ $\underbrace{\text{Chain see below}}$ (T)

Chain

inside $g(x) = e^{\sin(x^2)}$ $g'(x) = e^{\sin(x^2)}\cos(x^2)\,2x$

outside $f(u) = \sin u$ $f'(u) = \cos u$

$f \circ g \Rightarrow y$ ✓

$\frac{dy}{dx} = f'(g(x))g(x) = f'(e^{\sin(x^2)}) = \cos(e^{\sin x^2})\,e^{\sin(x^2)}\cos(x^2)\cdot 2x$

$y = (\sin x)^{\ln e^2}$ b/c $\ln e^2 = 2$

$= (\sin x)^2$

$g(x) = \sin(x)$ $g'(x) = \cos x$

$f(u) = u^2$ $f'(u) = 2u$

$\frac{dy}{dx} = f'(g(x))g'(x)$

$= 2\sin(x)\cos x$

$\underbrace{\text{chain see}}$ (☺)

inside $g(x) = \sin(x^2)$ $g'(x) = \cos(x^2)\cdot 2x$

$f(u) = e^u$ $f'(u) = e^u$

$f \circ g(x) = g(x)$ ✓ $g'(x) = f'(g(x))\cdot g(x) = e^{\sin(x^2)}\cos(x^2)2x$

(☺) inside $g(x) = x^2$ $g'(x) = 2x$

$f(u) = \sin(u)$ $f'(u) = \cos$

$f \circ g = g$ ✓ $g'(x) = f'(g(x))g'(x)$

7. Find the equations of all lines tangent to the curve described by the relation $x^2y^2+xy=2$ that are also parallel to the line described by $y=-x-\pi$.

we want to find all lines ($y=mx+b$) so that $m=-1$ and the lines are tangent to the curve $x^2y^2+xy=2$.

$m=$ slope of the line tangent to $x^2y^2+xy=2$ $= \left.\frac{dy}{dx}\right|_{\text{specified point}}$

we need dy/dx (*)

we need to find when $dy/dx=-1$

$$\frac{2xy^2+y}{-2x^2y-x}=-1$$

$\Rightarrow 2xy^2+y=\cancel{-}2x^2y+x$

$y(2xy+1)=+x(2xy+1)$

$\Rightarrow y(2xy+1)-x(2xy+1)=0$

$\Rightarrow (y-x)(2xy+1)=0$

$\Rightarrow y-x=0$ or $2xy+1=0$

$\Rightarrow x=y$ or $xy=-\frac{1}{2}$

if $xy=-1/2$ then $x^2y^2+xy=2$ would imply $(-\frac{1}{2})^2+(-\frac{1}{2})=2$ which is false

so it must be that $x=y$ then $x^2y^2+xy=2$ would imply $x^4+x^2=2$

$\Rightarrow x^4+x^2-2=0 \Rightarrow (x^2+2)(x^2-1)=0$

(*) finding dy/dx:

$\frac{d}{dx}(x^2y^2+xy)=\frac{d}{dx}(2)$

$\underbrace{\left[\frac{d}{dx}(x^2y^2)\right]}_{\text{product rule}}+\underbrace{\left[\frac{d}{dx}(xy)\right]}_{\text{product rule}}=0$

$\left[x^2\frac{d}{dx}(y^2)+\frac{d}{dx}(x^2)y^2\right]+\left[x\frac{d}{dx}(y)+\frac{d}{dx}(x)y\right]=$

$x^2 2y\frac{dy}{dx}+2xy^2+x\frac{dy}{dx}+y=0$

$-x^2 2y\frac{dy}{dx} \quad -x\frac{dy}{dx} \quad -x^2 2y\frac{dy}{dx} \quad -x\frac{dy}{dx}$

$2xy^2+y=-x^2 2y\frac{dy}{dx}-x\frac{dy}{dx}$

$2xy^2+y=(-x^2 2y-x)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}=\frac{2xy^2+y}{-2x^2y-x}$

$\Rightarrow x^2=-2$ or $x^2=1$

$\Rightarrow \cancel{x=\sqrt{-2}}$ or $x=1$ or -1

So the points with a tang. line // to $y=-x-\pi$ is $(1,1)$ and $(-1,-1)$

$\Rightarrow$ the line equations are

$y-1=-(x-1)$ or $y+1=-(x+1)$

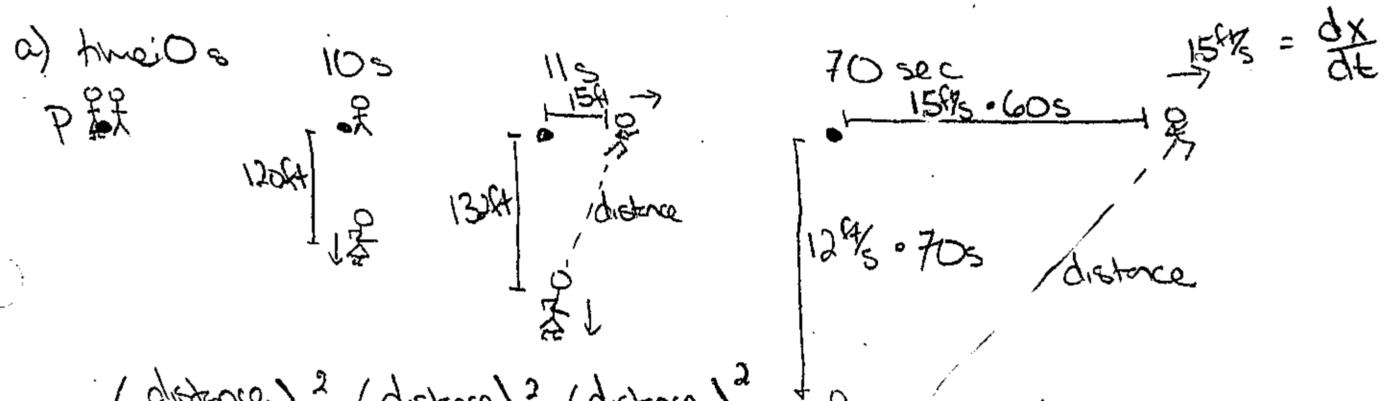
$\Rightarrow y=-x+2 \qquad y=-x-2$

9. [5] Choose *ONE* of the following. Clearly identify which of the two you are answering and what work you want to be considered for credit.
No, doing both questions will not earn you extra credit.

(a) (§3.9 #21) [5] Ryan and Stella were being chased by a pack of zombies. At point P they decided to split up and Stella ran south at 12 ft/s. Ryan waited for ten seconds to try to draw most of the zombies towards him and then started to run east at 15 ft/s. One minute later the two of them are still alive and running in their respective directions. At what rate are Ryan and Stella moving apart at this instant?

(b) A man walks along a straight path at a speed of 4ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15ft from the point on the path closest to the searchlight?

a)



$$\begin{pmatrix}\text{distance}\\ \text{between}\\ \text{them}\end{pmatrix}^2 = \begin{pmatrix}\text{distance}\\ \text{Ryan runs}\end{pmatrix}^2 + \begin{pmatrix}\text{distance}\\ \text{Stella runs}\end{pmatrix}^2$$

$\downarrow 12\text{ ft/s} = dy/dt$

$d^2 = x^2 + y^2$ where x = distance Ryan runs at time t
y = distance Stella runs at time t
d = distance between ryan + stella.

we want $\left.\dfrac{dd}{dt}\right|_{t=70}$

So $\frac{d}{dt}[d^2] = \frac{d}{dt}[x^2+y^2]$

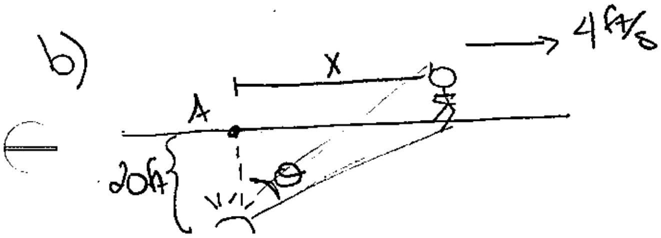
$\Rightarrow \dfrac{\cancel{2d}\,\frac{dd}{dt}}{\cancel{2d}} = \dfrac{2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt}}{2d}$

$\Rightarrow \dfrac{dd}{dt} = \dfrac{x\,\frac{dx}{dt} + y\,\frac{dy}{dt}}{d}$

when $t = 70$
$x = 15\,\text{ft}/\cancel{s} \cdot 60\cancel{s} = 900\text{ft}$
$y = 12\,\text{ft}/\cancel{s} \cdot 70\cancel{s} = 840\text{ft}$
Recall $d^2 = x^2 + y^2$
$\Rightarrow d = \sqrt{900^2 + 840^2}$

So $\left.\dfrac{dd}{dt}\right|_{t=70} = \dfrac{900(15) + 840(12)}{\sqrt{900^2 + 840^2}} \approx 19.2\text{ ft/s}$

b)



Let A be the point on the path closest to the searchlight.

Let x be the distance from the man to A

Let θ be the angle shown on the diagram.

we want to find $\left.\frac{d\theta}{dt}\right|_{x=15}$.

⇒ I need to find a relationship between θ and x

Then I'll differentiate with respect to t...

$$\tan\theta = \frac{x}{20}$$

$$\downarrow \tfrac{d}{dt}$$

$$(\sec^2\theta)\frac{d\theta}{dt} = \frac{d}{dt}\left(\frac{x}{20}\right)$$

$$\frac{d\theta}{dt}\left(\frac{1}{\cos^2\theta}\right) = \frac{d}{dt}\left(\frac{1}{20}\cdot x\right)$$

$$(\tan\theta)' = \left(\frac{\sin\theta}{\cos\theta}\right)' = \frac{\cos\theta(\sin\theta)' - \sin\theta(\cos\theta)'}{(\cos\theta)^2}$$

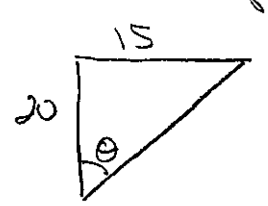
$$= \frac{\cos^2\theta - (\sin\theta)(-\sin\theta)}{\cos^2\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}$$

$$= 1/\cos^2\theta = \sec^2\theta$$

$$\frac{d\theta}{dt} = (\cos^2\theta)\frac{1}{20}\left(\frac{dx}{dt}\right) \qquad \text{since } dx/dt = 4$$

$$= (\cos^2\theta)\frac{1}{20}\cdot 4 = \frac{1}{5}\cos^2\theta = \frac{1}{5}(\cos\theta)^2$$

So we have a formula for $d\theta/dt$ + should be able to find $\left.\frac{d\theta}{dt}\right|_{x=15}$ but we need to find θ (or cos θ) when x = 15 to finish the computations...



$$\cos\theta = \frac{20}{hyp}$$

$$hyp^2 = 20^2 + 15^2 = 625$$

$$\Rightarrow hyp = 25$$

$$\cos\theta = \frac{20}{25}$$

$$\text{So } \left.\frac{d\theta}{dt}\right|_{x=15} = \frac{1}{5}\left(\frac{20}{25}\right)^2 = \frac{400}{3125} = \frac{\cancel{5}\cdot\cancel{5}\cdot 4\cdot 4}{\cancel{5}\cdot\cancel{5}\cdot 5\cdot 5\cdot 5} = \frac{16}{125} \text{ or } .128\ \frac{\text{radians}}{\text{second}}$$

10. [5] Choose *ONE* of the following. Clearly identify which of the two you are answering and what work you want to be considered for credit.
No, doing both questions will not earn you extra credit.

(a) (Story Problem Worksheet #10) If a current i passes through a resistor with resistance r, Ohm's Law states that the voltage drop is $v = ri$. Assume that voltage remains a constant 20 volts. An unreliable resistor claims a resistance of 10 ohms but may be off by up to 1.5 ohms. Use the linear approximation to approximate the error when calculating i.

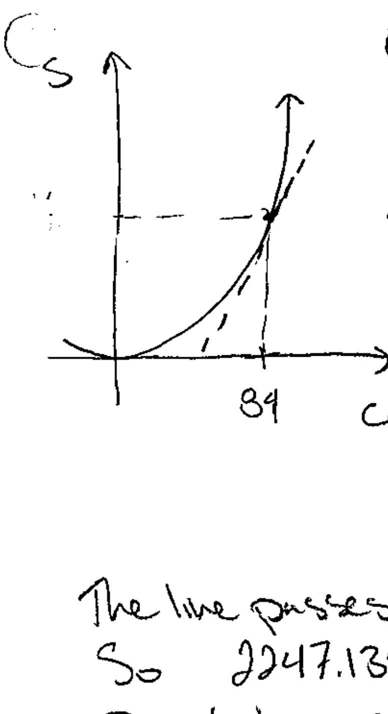
(b) (§3.10 #35) The circumference of a sphere was measured to be 84cm with a possible error of 0.5cm. Use linear approximation to find and upper bound and lower bound for the surface areas of this sphere.

b) I'd like to find a formula relating circumference of a sphere to the surface area. Then I'll use a line tangent to the graph of this formula to find my upper & lower bounds...

Recall: Let r be the radius of a sphere, C the circumference, and S the surface area. Then

$C = 2\pi r$ and $S = 4\pi r^2$

or $r = \frac{C}{2\pi}$ $\Rightarrow$ $= 4\pi\left(\frac{C}{2\pi}\right)^2 = \frac{\cancel{4\pi} C^2}{\cancel{4}\pi^{\cancel{2}}} = \frac{C^2}{\pi}$



Finding the equation of the line tangent to the graph when $C=84$

Looking for $y = mx + b$

m = slope of line tangent to graph S when $C = 84$

$= S'(84)$

Finding S': $\frac{d}{dC}(S) = \frac{d}{dC}\left(\frac{C^2}{\pi}\right) = \frac{d}{dC}\left(\frac{1}{\pi}C^2\right)$

$= \frac{1}{\pi}\frac{d}{dC}(C^2) = \frac{2}{\pi}\cdot C$

$= \frac{2\cdot 84}{\pi} = \frac{168}{\pi} \approx 53.503$

The line passes thru $(84, 84^2/\pi)$ or $(84, \frac{7056}{\pi}) \approx (84, 2247.134)$

So $2247.134 = 53.503(84) + b \Rightarrow b = -2247.118$

Dotted line equation is: $y = 53.503x - 2247.119$

An upper bound occurs if $C = 84 + \overset{\text{measurement error}}{.5} = 84.5$

So the upper bound of $S \approx 53.503\cdot(84.5) - 2247.119 \approx 2273.886c$

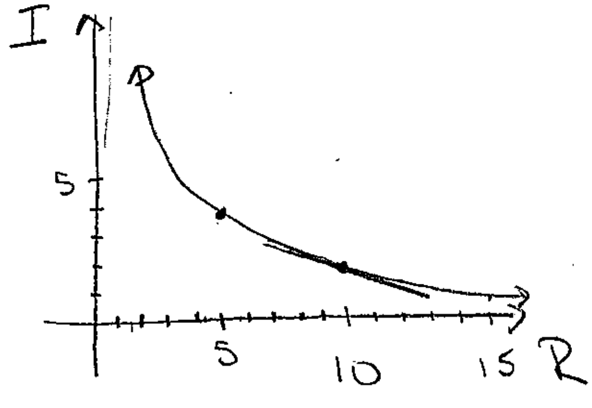
A lower bound occurs if $C = 84 - .5 = 83.5$

So the lower bound of $S \approx 53.503(83.5) - 2247.119 \approx 2220.383c$

a) Ohm's law $V = RI$ we'll want to examine I as a function of R

$V = 20$ volts

$$\Rightarrow I = \frac{V}{R} = \frac{20}{R} = 20R^{-1}$$



to approximate the error we'll use a linear approximation of the function $I = 20/R$

ie the line tangent to $I = 20/R$ when $R = 10$.

equation of line:

$y = mx + b$ or $y - y_1 = m(x - x_1)$

where y is I and x is R

m = slope of line tangent to $I = 20/R$ when $R = 10$

$$= I'\big|_{R=10}$$

$I' = 20(-1)R^{-2}$ by Power rule

$= -20R^{-2}$

$$\Rightarrow m = -20(10)^{-2} = \frac{-20}{10^2} = \frac{-2\cancel{0}}{10\cancel{0}} = \frac{-1}{5}$$

The line passes thru $(10, \frac{20}{10}) = (10, 2)$

So

$2 = -\frac{1}{5}(10) + b$ or $y - 2 = -\frac{1}{5}(x - 10)$

$\Rightarrow b = 2 + 2 = 4$ $\quad y - 2 = -\frac{1}{5}x + 2$

$I = -\frac{1}{5}R + 4$ $\quad I = -\frac{1}{5}R + 4$

If R is off by 1.5 ohms R could be anywhere between

$10 - 1.5 = 8.5$ or $10 + 1.5 = 11.5$

$\Rightarrow$ the approximate I values would thus range from

$-\frac{1}{5}(8.5) + 4$ to $-\frac{1}{5}(11.5) + 4$

or from 2.3 to 1.7

If R was exact I would be $\frac{20}{10} = 2$

Thus an approx bound on the error is

$|2.3 - 2| = |1.7 - 2| = .3$ amps

Warning: the graph suggests that we are underestimating the error!