

Key

Note: This is a practice final and is intended only for study purposes. The actual exam will contain different questions and may have a different layout.

1. [] TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F. Let f and g be differentiable functions and h be a constant.

T (F) $\frac{x+h}{2x} = \frac{1+h}{x}$ $\frac{1+h}{x} = \frac{2(1+h)}{2x} = \frac{2+2h}{2x}$

T (F) $\sqrt{x^2 + h^2} = x + h$ let $x=1$ and $h=1$ note $\sqrt{1^2 + 1^2} \neq 1+1$

(T) (F) $\lim_{x \rightarrow r} f(x) = f(r)$ for all r in the domain of f . *We were told that f is differentiable thus f is continuous*

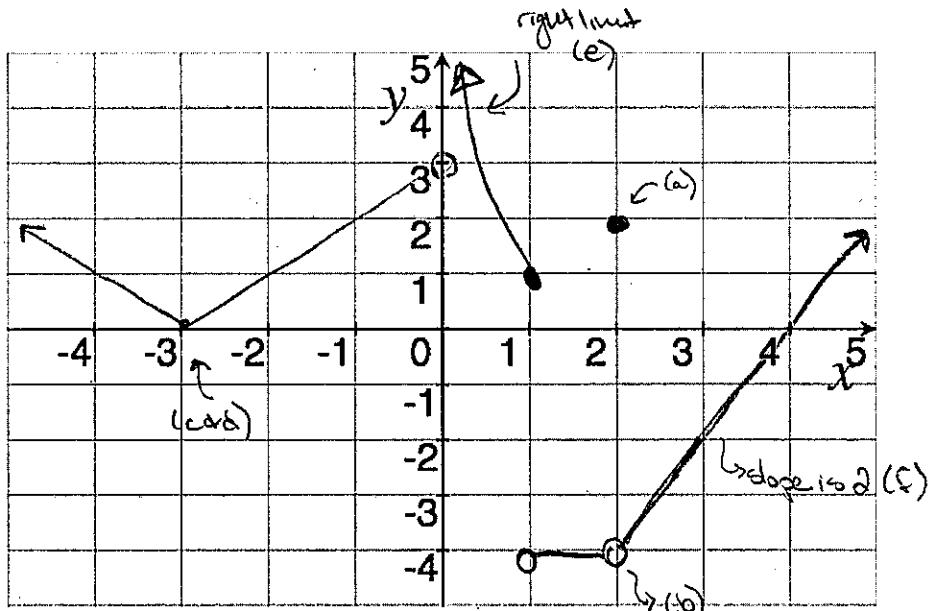
T (F) If $\lim_{x \rightarrow r} g(x) = 0$, then $\lim_{x \rightarrow r} \frac{f(x)}{g(x)}$ does not exist.

T (F) $\frac{d}{dx}\left(\frac{1}{x}\right) = -1$ $\frac{d}{dx}(x^{-1}) = -x^{-2} = -\frac{1}{x^2}$

Show your work for the following problems. The correct answer with no supporting work will receive NO credit (this includes multiple choice questions).

2. [] Sketch the graph and then find the formula of an example function f that satisfies the following conditions:

- (a) $f(2) = 2$
- (b) $\lim_{x \rightarrow 2^-} f(x) = -4$
- (c) f is not differentiable when $x = -3$
- (d) f is continuous when $x = -3$
- (e) $\lim_{x \rightarrow 0^+} f(x) = \infty$
- (f) $f'(4) = 2$



$$\begin{cases} x+3 & \text{if } x < 0 \\ \frac{1}{x} & \text{if } 0 < x \leq 1 \\ -4 & \text{if } 1 < x \leq 2 \\ 2 & \text{if } x = 2 \\ 2x-3 & \text{if } 2 < x \end{cases}$$

3. Compute the following limits:

$$(a) \lim_{x \rightarrow 1} \frac{x^2 + x - 2}{2x^2 - 8x + 6} \quad \left[\begin{matrix} \text{den} = 0 \\ 2-8+6=0 \end{matrix} \right]$$

$$= \lim_{x \rightarrow 1} \frac{(x+2)(x-1)}{2(x-3)(x-1)}$$

$$= \lim_{x \rightarrow 1} \frac{x+2}{2(x-3)} = \frac{1+2}{2(1-3)} = \frac{3}{-4}$$

$$= \frac{3}{-4}$$

$$(c) \lim_{\theta \rightarrow 0^+} \frac{\theta + \theta^2}{1 - \cos \theta} \quad \frac{0+0}{1-1} = "0"$$

$$\text{L'H} = \lim_{\theta \rightarrow 0^+} \frac{1+2\theta}{-(\sin \theta)}$$

$$= \lim_{\theta \rightarrow 0^+} \frac{1+2\theta}{\sin \theta} \quad \left[\begin{matrix} \text{den} = 0 \\ \sin 0 = 0 \end{matrix} \right]$$

looks like " $\frac{1}{0}$ " but b/c $\theta \rightarrow 0^+$

$\sin \theta$ is + \Rightarrow limit is $+\infty$

$$(e) \lim_{x \rightarrow 0} x^4 \sin\left(\frac{1}{x}\right)$$

note $\sin \frac{1}{x}$ as $x \rightarrow 0$ never 'settles down' but for all x

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

\Rightarrow if we mult the inequalities by x^4

$$-x^4 \leq x^4 \sin\left(\frac{1}{x}\right) \leq x^4$$

$$\text{Observe } \lim_{x \rightarrow 0} -x^4 = 0 = \lim_{x \rightarrow 0} x^4$$

so by the squeeze theorem

$$\lim_{x \rightarrow 0} x^4 \sin\left(\frac{1}{x}\right) = 0.$$

$$(b) \lim_{x \rightarrow \infty} \frac{x^2 + x - 2}{2x^2 - 8x + 6} \quad \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$\lim_{x \rightarrow \infty} \frac{\frac{x^2}{x^2} + \frac{x}{x^2} - \frac{2}{x^2}}{2 - \frac{8}{x^2} + \frac{6}{x^2}} = \lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x} - \frac{2}{x^2}}{2 - \frac{8}{x^2} + \frac{6}{x^2}}$$

$$= \frac{\lim_{x \rightarrow \infty} (1 + \frac{1}{x} - \frac{2}{x^2})}{\lim_{x \rightarrow \infty} (2 - \frac{8}{x^2} + \frac{6}{x^2})} = \frac{1}{2}$$

$$(d) \lim_{x \rightarrow \infty} x \sin\left(\frac{5\pi}{x}\right) = \lim_{x \rightarrow \infty} \frac{\sin\left(\frac{5\pi}{x}\right)}{\frac{1}{x}} = \frac{0}{0}$$

$$\text{L'H} = \lim_{x \rightarrow \infty} \frac{\cos\left(\frac{5\pi}{x}\right) \cdot -\frac{5\pi}{x^2}}{-\frac{1}{x^2}}$$

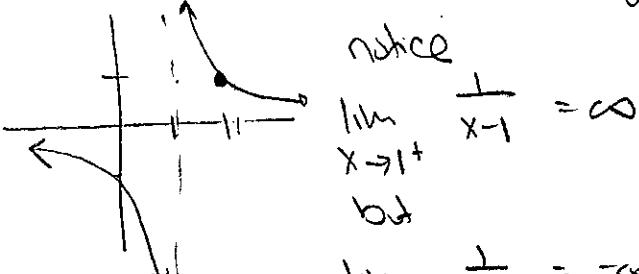
$$= \lim_{x \rightarrow \infty} \left[\cos\left(\frac{5\pi}{x}\right) \cdot -\frac{5\pi}{x^2} \right] \div \left[\frac{1}{x^2} \right]$$

$$= \lim_{x \rightarrow \infty} 5\pi \cos\left(\frac{5\pi}{x}\right) = 5\pi \lim_{x \rightarrow \infty} \cos\left(\frac{5\pi}{x}\right)$$

$$(f) \lim_{x \rightarrow 1} \frac{1}{x-1} \quad \left[\begin{matrix} \text{den} = 0 \\ 1-1=0 \end{matrix} \right]$$

note $\frac{1}{x-1}$ looks like the graph

of $\frac{1}{x}$ shifted horiz. to the right 1 unit



notice

$$\lim_{x \rightarrow 1^+} \frac{1}{x-1} = \infty$$

but

$$\lim_{x \rightarrow 1^-} \frac{1}{x-1} = -\infty$$

thus $\lim_{x \rightarrow 1} \frac{1}{x-1}$ doesn't exist.

4. Let $f(x) = \begin{cases} \sqrt{1-(x+3)^2} & \text{if } -4 \leq x \leq -2 \\ 1 & \text{if } -2 < x < 1 \\ -(x-2)^2 + 2 & \text{if } 1 < x \end{cases}$

$y = \sqrt{1-(x+3)^2} \Rightarrow y^2 + (x+3)^2 = 1$ circle centred at $(-3, 0)$ with radius 1
 parabola with vertex $(2, 2)$ opening downwards

Graph $f(x)$ and then sketch the graph $f'(x)$ below on its own set of axes. Afterwards, answer the following questions.

(a) $\lim_{x \rightarrow 1} f(x)$

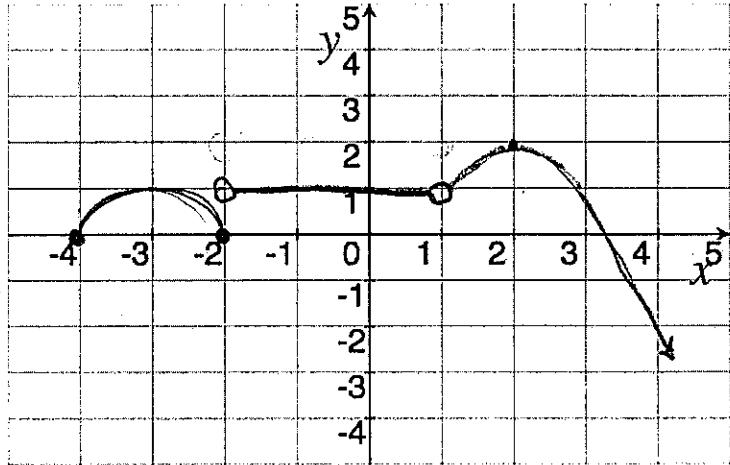
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(b) $\lim_{x \rightarrow 3} [4f(x) - 7]$

$4 \lim_{x \rightarrow 3} f(x) - 7 = 4 \cdot 1 - 7 = -3$

(c) $\lim_{x \rightarrow -2} f(x)$

DNE



(d) $\lim_{x \rightarrow -2^-} f(x)$

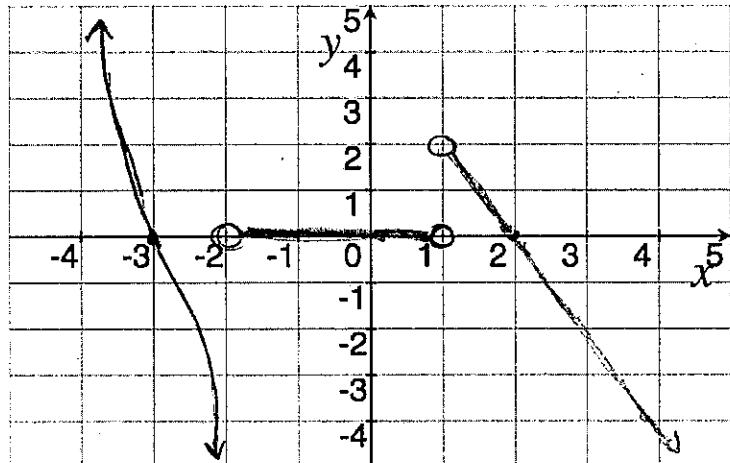
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(e) $\lim_{x \rightarrow 3} f'(x)$

-2 (using graph of f')

(f) $\lim_{x \rightarrow \infty} f(x)$

$-\infty$



(g) $[f + f]'(2)$

$= f'(2) + f'(2)$

$= 0 + 0$

= 0

note
 $y^2 + (x+3)^2 = 1$
 $\frac{dy}{dx}(y^2 + (x+3)^2) = \frac{dy}{dx}(1)$
 $2y \frac{dy}{dx} + 2(x+3) = 0$
 $\frac{dy}{dx} = -\frac{2(x+3)}{2y}$
 $y' = \frac{-(x+3)}{\sqrt{1-(x+3)^2}}$

note
 $\frac{d}{dx}(-(x-2)^2 + 2)$
 $\frac{d}{dx}(-x^2 + 4x - 2)$
 $-2x + 4$

5. Compute the derivatives of the following functions. You do *not* need to simplify.

$$(a) f(x) = x^3 + 3^x + \pi^\pi$$

$$f'(x) = 3x^2 + 3^x \ln 3 + 0$$

$$(b) g(t) = \ln(t) \left(\frac{2+t^2}{3t-1} \right)$$

$$g'(t) = \ln(t) \left[\frac{2+t^2}{3t-1} \right]' + [\ln(t)]' \left(\frac{2+t^2}{3t-1} \right)$$

$$= \ln(t) \left[\frac{(3t-1)(2t) - (2+t^2)(3)}{(3t-1)^2} \right] + \frac{1}{t} \left(\frac{2+t^2}{3t-1} \right)$$

$$(c) h(\theta) = 7 \sec(\sqrt{\theta})$$

$$h'(\theta) = 7 \left[(\cos(\theta^6))^{-1} \right]'$$

$$= 7 \cdot -1 (\cos(\theta^6))^{-2} \cdot (-\sin(\theta^6)) \frac{1}{2} \theta^{\frac{1}{2}}$$

$$= 7 \frac{\sin \sqrt{\theta}}{(\cos \sqrt{\theta})^2} \frac{1}{2} \frac{1}{\sqrt{\theta}} = \frac{7 \sin \sqrt{\theta}}{2\sqrt{\theta} (\cos \sqrt{\theta})^2}$$

$$(d) y = \sqrt{x} e^{x^7} (x^6 + 3)^{10} \quad \ln y = \ln \left[x^{\frac{1}{2}} e^{x^7} (x^6 + 3)^{10} \right]$$

$$\ln y = \frac{1}{2} \ln x + x^7 + 10 \ln(x^6 + 3) \quad \frac{dy}{dx}$$

$$\frac{1}{y} y' = \frac{1}{2} \cdot \frac{1}{x} + 7x^6 + \frac{10}{x^6 + 3} \cdot 6x^5$$

$$y' = y \left[\frac{1}{2x} + 7x^6 + \frac{60x^5}{x^6 + 3} \right]$$

$$(c) y = (\cos(x))^x$$

$$\ln y = \ln(\cos(x))^x$$

$$\ln y = x \ln(\cos(x))$$

$$(d) x^2 y^2 = 4 - y \arctan(5x) \quad \text{note: } [\arctan(x)]' = \frac{1}{1+x^2}$$

$$x^2 \frac{d}{dx} y y' + 2xy^2 = 0 - [y \frac{1}{1+(5x)^2} \cdot 5 + y' \arctan 5x]$$

$$2x^2 y y' + 2xy^2 = -\frac{5y}{1+25x^2} - y' \arctan 5x$$

$$2x^2 y y' + y' \arctan 5x = -2xy^2 - \frac{5x}{1+25x^2}$$

$$y' (2x^2 y + \arctan 5x) = " "$$

$$4 \quad y' = \frac{-5x}{1+25x^2 - 2xy^2}$$

$$2x^2 y + \arctan(5x)$$

6. Find the equation of the line tangent to the graph of f when $x = 2$ if $f(x) = m(n(x))$, $n(2) = -1$, $m(-1) = 6$, $n'(2) = 3$, and $m'(-1) = 5$.

Looking for $y = mx + b$

$$m = f'(2)$$

$$\begin{aligned} f'(x) &= (m \circ n)'(x) \text{ Chain Rule?} \\ &= m'(n(x)) n'(x) \end{aligned}$$

$$\text{So } f'(2) = m'(n(2)) n'(2)$$

$$= m'(-1) \cdot 3 = 5 \cdot 3 = 15$$

$$\rightarrow \text{So we have } y = 15x + b.$$

$$\begin{aligned} \text{Line passes through } (2, f(2)) \\ \text{or } (2, m(n(2))) = (2, m(-1)) = (2, 6) \\ \text{so} \end{aligned}$$

$$6 = 15 \cdot 2 + b \rightarrow b = 6 - 30 = -24$$

Thus

$$y = 15x - 24$$

7. Find the antiderivative for each of the following functions:

$$(a) 2x - x^3 + 7 \sin(x)$$

$$x^2 - \frac{1}{4}x^4 + 7 \cos(x)$$

check:

$$(x^2 - \frac{1}{4}x^4 + 7 \cos(x))' = 2x - \frac{1}{4} \cdot 4x^3 + 7(-\sin(x))$$

off by negative sign in last term so

$$\boxed{x^2 - \frac{1}{4}x^4 - 7 \cos(x)}$$

$$\text{by } x^{-5} + x^{-2} + 2x$$

check:

$$(x^{-5} + x^{-2} + 2x)' = -5x^{-6} - 2x^{-3} + 2$$

off by neg sign & a factor of 2.

$$\boxed{-x^{-5} + 2x^{-2} + 2x}$$

8. Consider the function $f(x) = \sqrt[3]{x}$

$$(a) \text{ Evaluate the integral } \int_1^8 \sqrt[3]{x} dx = F(8) - F(1) \text{ where } F \text{ is an antiderivative}$$

note

$$(\frac{3}{4}x^{\frac{4}{3}})' = \frac{3}{4} \cdot \frac{4}{3} x^{\frac{1}{3}} = x^{\frac{1}{3}} \text{ so } \frac{3}{4}x^{\frac{4}{3}}$$

is an antiderivative

$$\begin{aligned} \int_1^8 x^{\frac{1}{3}} dx &= \frac{3}{4} x^{\frac{4}{3}} \Big|_1^8 \\ &= \frac{3}{4}(8)^{\frac{4}{3}} - \frac{3}{4}(1)^{\frac{4}{3}} \end{aligned}$$

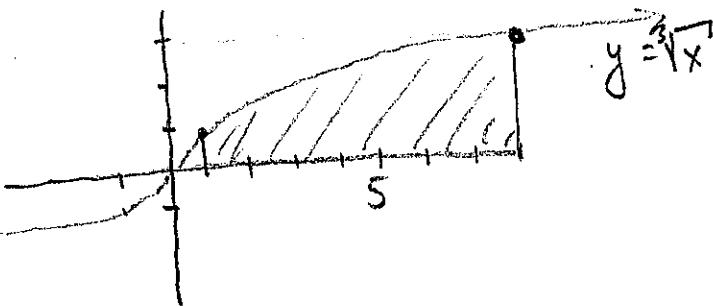
- (b) Draw a picture that corresponds to the area you computed in (a).

$$= \frac{3}{4} \cdot 2^4 - \frac{3}{4}$$

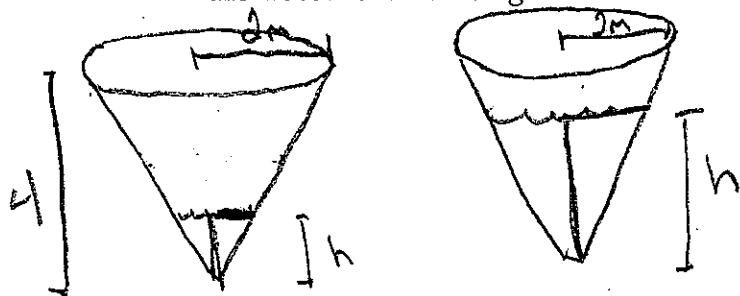
$$= 12 - \frac{3}{4}$$

$$= \frac{48 - 3}{4}$$

$$= \frac{45}{4}$$



9. A water tank has the shape of an inverted circular cone with base radius 2m and height 4m. If water is being pumped into the tank at a rate of $2\text{m}^3/\text{min}$, find the rate at which the water level is rising when the water is 3m deep.



want to find $\frac{dh}{dt}$ |
h = 3m
know $\frac{dV}{dt} = 2 \text{ m}^3/\text{min}$.

need relation between V & h

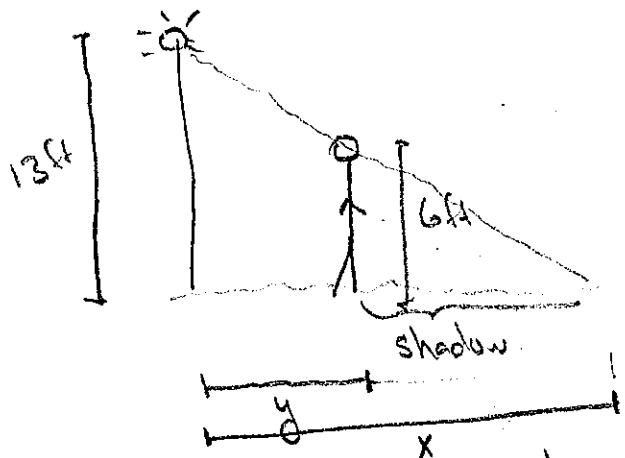
V is volume of water The volume of a cone is $\frac{1}{3}\pi(\text{radius})^2 \cdot \text{height}$.

$V = \frac{1}{3}\pi r^2 h$ this would be easier to take the derivative if we could have it in terms of only 1 variable.

Note, similar Δ so $V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 \cdot h \Rightarrow \frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2 \cdot \frac{dh}{dt}$

$\Rightarrow \frac{r}{h} = \frac{4}{n} \Rightarrow r = \frac{2h}{3}$ $V = \frac{1}{12}\pi h^3 \Rightarrow \text{when } h = 3 \Rightarrow \frac{d}{dt} = \frac{\pi}{12} \cdot 3 \cdot 3^2 \cdot \frac{dh}{dt}$

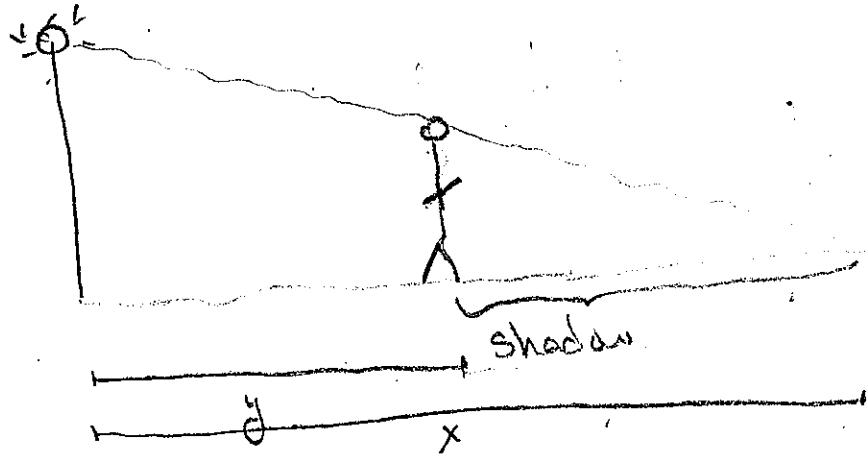
10. A street light is mounted at the top of a 13 ft pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 30 ft away from the pole?



$$\frac{dy}{dt} = 5 \text{ ft/s} \text{ want } \frac{dx}{dt}$$

Note similar Δ 's.

$$\frac{13}{x} = \frac{6}{x+y} \Rightarrow 13x - 13y = 6x \Rightarrow 7x = 13y$$

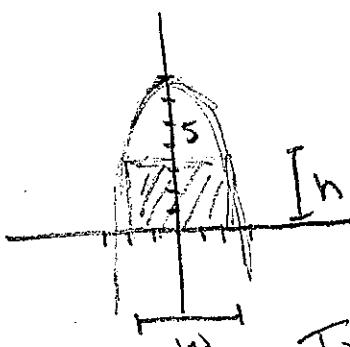


$$7 \frac{dx}{dt} = 13, \frac{dy}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{13}{7} \frac{dy}{dt}$$

so when 30ft from the pole
the tip of the shadow is
moving at $\frac{13}{7} \cdot 5 = \frac{65}{7} \text{ ft/s}$

11. Find the dimensions of the rectangle of largest area that has its base on the x -axis and its other two vertices above the x -axis and lying on the parabola $y = 7 - x^2$



want to maximize area = $w \cdot h$

$$\text{note } w = 2x$$

$$\text{and } h = y = 7 - x^2$$

$$\text{so area} = 2x(7 - x^2) = 14x - 2x^3$$

To maximize we need to find the extrema.

$$\begin{aligned} \text{Area}' &= 14 - 6x^2 \Rightarrow \frac{-14}{-6} = x^2 \quad \text{Area}(G) > \text{Area}(C) \quad \text{Max when} \\ 0 &= 14 - 6x^2 \qquad \qquad \qquad x = \sqrt{\frac{7}{3}} \\ -14 &= -6x^2 \qquad \qquad \qquad \text{so width} = 2\sqrt{\frac{7}{3}} \end{aligned}$$

12. A truck has a minimum speed of 9 mph in high gear. When traveling x mph, the truck burns diesel fuel at the rate of

$$0.003935 \left(\frac{675}{x} + x \right) \frac{\text{gal}}{\text{mile}}$$

Assume that the truck can not be driven over 63 mph, that diesel fuel costs \$2.84 a gallon, and that the driver is paid \$12 an hour. Find the speed that will minimize the cost of a 500 mile trip.

Total Cost = Cost of Gas + Cost of Driver.

$$\begin{aligned} &= 5.5877 \left(\frac{675}{x} + x \right) + \frac{6000}{x} \\ &= \frac{3771.6975}{x} + 5.5877x + \frac{6000}{x} \\ &= \frac{9771.6975}{x} + 5.5877x \end{aligned}$$

$$\text{Total cost}'(x) = -\frac{9771.6975}{x^2} + 5.5877$$

Cost of Gas:

$$\begin{aligned} &0.003935 \left(\frac{675}{x} + x \right) \frac{500 \text{ miles}}{\text{miles}} \cdot 2.84 \frac{\$}{\text{gal}} \\ &= 5.5877 \left(\frac{675}{x} + x \right) \text{ dollars} \end{aligned}$$

Cost of Driver:

$$\begin{aligned} &12 \frac{\$}{\text{hr}} \cdot \frac{x}{x \text{ miles}} \cdot 500 \text{ miles} \\ &= \frac{6000}{x} \text{ dollars} \end{aligned}$$

Note $x = 41.82 \text{ mph}$ won't work
Verify $x = 41.8 \text{ mph}$ is a min.

$$\Rightarrow 5.5877x^2 = 9771.6975$$

$$\Rightarrow x = \pm 41.82 \text{ mph}$$

$$\frac{-}{\text{Cost}'(1) \ 41.8 \ \text{Cost}'(50)} +$$

$$\text{min} \rightarrow \rightarrow \rightarrow 41.8 \text{ mph}$$