EXAM 2

TMath 124

Practice

Key

Note: This is a practice midterm and is intended only for study purposes. The actual exam will contain different questions and perhaps a different layout.

1. [] TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F. Let f and g be functions.

   T F $\frac{d}{dx}b^c = cb^{c-1}$ for a fixed b and c $\quad b^c$ is a constant so $\frac{d}{dx}(b^c) = 0$

   T F $(x+y)^2 = x^2 + y^2$ $\quad (x+y)^2 = (x+y)(x+y) = x^2 + xy + xy + y^2$

   T F $\frac{d}{dx}2^x = x2^{x-1}$ $\quad \frac{d}{dx}(2^x) = 2^x(\ln 2)$

Show your work for the following problems. The correct answer with no supporting work will receive NO credit (this includes multiple choice questions).

2. [] Sketch the graph of an example function f that satisfies the following conditions:

   (a) f is not differentiable when $x = -3$ ✓

   (b) f is continuous when $x = -3$ ✓

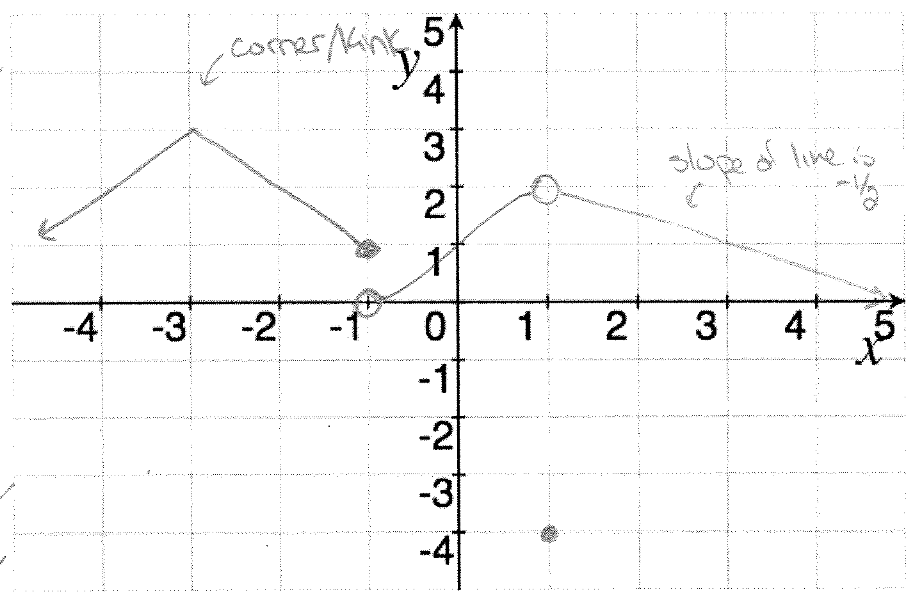
   (c) $f(1) = -4$ ✓

   (d) $\lim_{x\to 1} f(x) = 2$ ✓

   (e) $f'(3) = -\frac{1}{2}$ ✓

   (f) $\lim_{x\to -1^+} f(x) = 0$ ✓

   (g) $\lim_{x\to -1^-} f(x) = 1$ ✓



Find a formula for the above graph

$$f(x) = \begin{cases} x+6 & \text{if } x \le -3 \\ -x & \text{if } -3 \le x \le -1 \\ x+1 & \text{if } -1 < x < 1 \\ 4 & \text{if } x = 1 \\ -\frac{1}{2}x + 2.5 & \text{if } 1 < x \end{cases}$$

1

$\sin^2 x + \cos^2 x = 1$

$\Rightarrow \cos^2 x - 1 = -\sin^2 x$

3. Find the following:

$$\lim_{x\to 0} \frac{3\sin(4x)}{2\sin(3x)} \cdot \frac{2x}{2x} = \lim_{x\to 0} \left(\frac{\sin 4x}{4x}\right)\left(\frac{3\cdot 2x}{\sin 3x}\right)$$

$$= \lim_{x\to 0} \frac{\sin 4x}{4x} \cdot \lim_{x\to 0} \left(\frac{3x}{\sin 3x}\right)\cdot 2$$

$$= \lim_{x\to 0} \frac{1}{\frac{\sin 3x}{3x}} \lim_{x\to 0} 2 = 2$$

$$\lim_{x\to 0} \frac{\cos x - 1}{\sin x} \frac{(\cos x + 1)}{(\cos x + 1)}$$

$$= \lim_{x\to 0} \frac{\cos^2 x - 1}{\sin(x)(\cos(x)+1)}$$

$$= \lim_{x\to 0} \frac{-\sin^{\cancel{2}} x}{\cancel{\sin(x)}(\cos(x)+1)} = \lim_{x\to 0} \frac{-\sin x}{\cos(x)+1}$$

$$= \frac{-0}{2} = 0$$

4. Suppose that $f(2) = -3$, $g(2) = 4$, $f'(2) = -2$, and $g'(2) = 7$. Find $h'(2)$ where h is:

$h(x) = 5f(x) - 4g(x)$

$$h'(x) = [5f(x) - 4g(x)]'$$
$$= [5f(x)]' - [4g(x)]'$$
$$= 5f'(x) - 4g'(x)$$
$$\Rightarrow h'(2) = 5f'(2) - 4g'(2)$$
$$= 5(-2) - 4(7) = -10 - 28 = -38$$

$h(x) = f(x)g(x)$ product rule

$$h'(x) = [f(x)g(x)]' = f(x)g'(x) + f'(x)g(x)$$
$$\Rightarrow h'(2) = f(2)g'(2) + f'(2)g(2)$$
$$= (-3)(7) + (-2)(4)$$
$$= -21 - 8 = -29$$

$h(x) = \frac{f(x)}{g(x)}$ quotient rule

$$h'(x) = \left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\Rightarrow h'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{[g(2)]^2} = \frac{(4)(-2) - (-3)(7)}{4^2}$$

$$= (-8+21)/16 = 13/16$$

$h(x) = \frac{g(x)}{1+f(x)}$ quotient rule

$$h'(x) = \left[\frac{g(x)}{1+f(x)}\right]' = \frac{[1+f(x)]g'(x) - g(x)[1+f(x)]'}{[1+f(x)]^2}$$

$$\Rightarrow h'(2) = \frac{[1+f(2)]g'(2) - g(2)f'(2)}{[1+f(2)]^2}$$

$$= \frac{[1+(-3)](7) - 4(-2)}{[1+(-3)]^2} = \frac{-14+8}{4} = \frac{-3}{2}$$

5. If $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, and $g'(5) = 6$, find $F'(5)$.

$F'(x) = f'(g(x))\,g'(x)$ by chain rule

$F'(5) = f'(g(5))g'(5) = f'(-2)\cdot 6 = 4\cdot 6 = 24$

note: $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x(\sin x)' - \sin x(\cos x)'}{(\cos x)^2}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$ ✓

6. Find the $\frac{dy}{dx}$ of the following:

$$y = (2x^2 + 7x^2)(e^x - 2^x)$$ product rule

$= (9x^2)(3^x - 2^x)$

$dy/dx = (9x^2)\frac{d}{dx}(3^x - 2^x) + \frac{d}{dx}(9x^2)(3^x - 2^x)$

$= 9x^2\left[\frac{d}{dx}(3^x) - \frac{d}{dx}(2^x)\right] + 18x(3^x - 2^x)$

$= 9x^2[3^x(\ln 3) - 2^x(\ln 2)] + 18x(e^x - 2^x)$

$$y = \frac{\sin(x) + x^2\cos(x)}{\cos(x)}$$

$= \frac{\sin(x)}{\cos(x)} + \frac{x^2\cos(x)}{\cos x}$ note: domain remains

$= \tan(x) + x^2$

$dy/dx = \frac{d}{dx}[\tan(x) + x^2]$

$= \frac{d}{dx}[\tan(x)] + \frac{d}{dx}(x^2)$

$= \sec^2 x + 2x$

$$y = \sqrt{\frac{x^2+1}{4x^5 - 3x}}$$ quotient rule see below ✗

Chain

inside $g(x) = \frac{x^2+1}{4x^5-3x}$ $g'(x) =$

outside $f(u) = u^{1/2}$ $f'(u) = \frac{1}{2}u^{-1/2}$

$f \circ g(x)$ ✓

$dy/dx = f'(g(x))g'(x) = \frac{1}{2}\left(\frac{x^2+1}{4x^5-3x}\right)^{-1/2}\left[\frac{(4x^5-3x)2x - (x^2+1)(20x^4-3)}{[4x^5-3x]^2}\right]$

✗ $g'(x) = \frac{(4x^5-3x)[x^2+1]' - [x^2+1](4x^5-3x)'}{[4x^5-3x]^2}$

$= \frac{(4x^5-3x)\,2x - (x^2+1)(20x^4-3)}{[4x^5-3x]^2}$

$$e^y\sin(x) = x + xy$$

$\frac{d}{dx}(e^y\sin(x)) = \frac{d}{dx}(x + xy)$

product rule

$e^y\frac{d}{dx}(\sin x) + \frac{d}{dx}(e^y)\sin x = \frac{d}{dx}(x) + \frac{d}{dx}(xy)$

$e^y\cos x + e^y y'\sin x = 1 + \frac{d}{dx}(xy)$ product rule

$e^y\cos x + y'e^y\sin x = 1 + x\frac{d}{dx}(y) + \frac{d}{dx}(x)y$

$e^y\cos x + y'e^y\sin x = 1 + xy' + y$

$-e^y\cos x \quad -xy' \quad -xy' \quad -e^y\cos x$

$y' = \frac{1 + y - e^y\cos x}{e^y\sin x - x}$

$$y = \sin(e^{\sin(x^2)})$$ chain see below (T)

Chain

inside $g(x) = e^{\sin(x^2)}$ $g'(x) = e^{\sin(x^2)}\cos(x^2)2x$

outside $f(u) = \sin u$ $f'(u) = \cos u$

$f \circ g = y$

$dy/dx = f'(g(x))g'(x) = f'(e^{\sin(x^2)}) = \cos(e^{\sin x^2})\,e^{\sin(x^2)}\cos(x^2)\cdot 2x$

$$y = (\sin x)^{\ln e^2}$$ b/c $\ln e^2 = 2$

$= (\sin x)^2$

$g(x) = \sin(x)$ $g'(x) = \cos x$

$f(u) = u^2$ $f'(u) = 2u$

$dy/dx = f'(g(x))g'(x)$

$= 2\sin(x)\cos x$

chain see (T)

inside $g(x) = \sin(x^2)$ $g'(x) = \cos(x^2)\cdot 2x$

$f(u) = e^u$ $f'(u) = e^u$

$f \circ g(x) = g(x)$ ✓ $g'(x) = f'(g(x))\cdot g(x) = e^{\sin(x^2)}\cdot\cos(x^2)2x$

(T) inside $g(x) = x^2$ $g'(x) = 2x$

$f(u) = \sin(u)$ $f'(u) = \cos$

$f \circ g = g$ ✓ $g'(x) = f'(g(x))\,g'(x)$

7. Find the equations of all lines tangent to the curve described by the relation $x^2y^2+xy = 2$ that are also parallel to the line described by $y = -x - \pi$.

we want to find all lines $(y=mx+b)$ so that $m=-1$ and the lines are tangent to the curve $x^2y^2+xy=2$.

$m =$ slope of the line tangent to $x^2y^2+xy=2$ $= \left.\frac{dy}{dx}\right|_{\text{specified point}}$

we need $\frac{dy}{dx}$ (*)

we need to find when $\frac{dy}{dx}=-1$

$$\frac{2xy^2+y}{-2x^2y-x} = -1$$

$\Rightarrow 2xy^2+y = 2x^2y+x$

$y(2xy+1) = x(2xy+1)$

$\Rightarrow y(2xy+1) - x(2xy+1) = 0$

$\Rightarrow (y-x)(2xy+1)=0$

$\Rightarrow y-x=0$ or $2xy+1=0$

$\Rightarrow x=y$ or $xy=-\frac{1}{2}$

if $xy=-\frac{1}{2}$ then $x^2y^2+xy=2$ would imply $(-\frac{1}{2})^2+(-\frac{1}{2})=2$ which is false

so it must be that $x=y$ then $x^2y^2+xy=2$ would imply $x^4+x^2=2$

$\Rightarrow x^4+x^2-2=0 \Rightarrow (x^2+2)(x^2-1)=0$

(*) finding $\frac{dy}{dx}$:

$\frac{d}{dx}(x^2y^2+xy) = \frac{d}{dx}(2)$

$\underbrace{\left[\frac{d}{dx}(x^2y^2)\right]}_{\text{product rule}} + \underbrace{\left[\frac{d}{dx}(xy)\right]}_{\text{product rule}} = 0$

$\left[x^2\frac{d}{dx}(y^2) + \frac{d}{dx}(x^2)y^2\right] + \left[x\frac{d}{dx}(y) + \frac{d}{dx}(x)y\right] = 0$

$x^2 \cdot 2y\frac{dy}{dx} + 2xy^2 + x\frac{dy}{dx} + y = 0$

$2xy^2+y = -x^2 2y\frac{dy}{dx} - x\frac{dy}{dx}$

$2xy^2+y = (-x^2 2y - x)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{2xy^2+y}{-2x^2y-x}$

$\Rightarrow x^2=-2$ or $x^2=1$

$\Rightarrow$ ~~$x=\sqrt{-2}$~~ or $x=1$ or -1

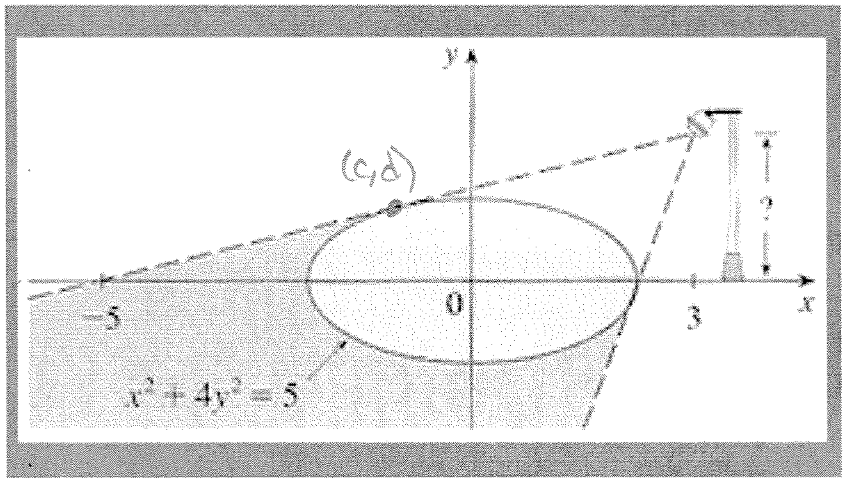
So the points with a tang line // to $y=-x-\pi$ is $(1,1)$ and $(-1,-1)$

$\Rightarrow$ the line equations are

$y-1=-(x-1)$ or $y+1=-(x+1)$

$\Rightarrow y=-x+2$ $\quad y=-x-2$

8. [10] (§3.6 #69) The figure below shows a lamp located three units to the right of the y-axis and a shadow created by the elliptical region $x^2 + 4y^2 \le 5$. The point $(-5, 0)$ is on the edge of the shadow.



(a) [2] Find $\frac{dy}{dx}$ of the ellipse.

equation of an ellipse : $x^2+4y^2=5$

$\frac{d}{dx}(x^2+4y^2) = \frac{d}{dx}(5)$

$\frac{d}{dx}(x^2) + \frac{d}{dx}(4y^2) = 0$

$2x + 4\frac{d}{dx}(y^2) = 0$

$2x + 4 \cdot 2y \, dy/dx = 0$ → $\frac{8y \, dy/dx}{8y} = \frac{-2x}{8y}$ ⇒ $dy/dx = \frac{-x}{4y}$

(b) [3] Denote the point that is both on the ellipse and the top dashed line by (c, d). Notice that the slope of the top dashed line is thus $\frac{d-0}{c-(-5)}$. Use this information and what you found in part (a) to find the value of c.

m = slope of line tang to ellipse at (c,d) $= \left.\frac{dy}{dx}\right|_{(c,d)}$

$\frac{d-0}{c-(-5)} = \frac{-c}{4d}$ ⇒ $\frac{d}{c+5} = \frac{-c}{4d}$ → $d(4d) = -c(c+5)$

$4d^2 = -c^2-5c$ ✱

b/c (c,d) is a point on the ellipse we know $c^2+4d^2=5$ or $4d^2 = 5-c^2$

Sub into ✱ $5-c^2 = -c^2-5c$

⇒ $5 = -5c$

⇒ $-1 = c$

(c) [5] Find the equation of the top dashed line and then find out the height of the lamp. Looking for $y = mx+b$

when $c=-1$ d must be $(-1)^2 + 4(\ ^2) = 5$

⇒ $4d^2 = 4$

⇒ $d^2 = 1$

⇒ $d = 1$ or -1

we're looking at the point with $d>0$ so the point on the dashed line is $(-1,1)$

m = slope of line tangent to $x^2+4y^2=5$ at $(-1,1)$

$= \left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{-(-1)}{4(1)} = \frac{1}{4}$

So $y - 1 = \frac{1}{4}(x-(-1))$

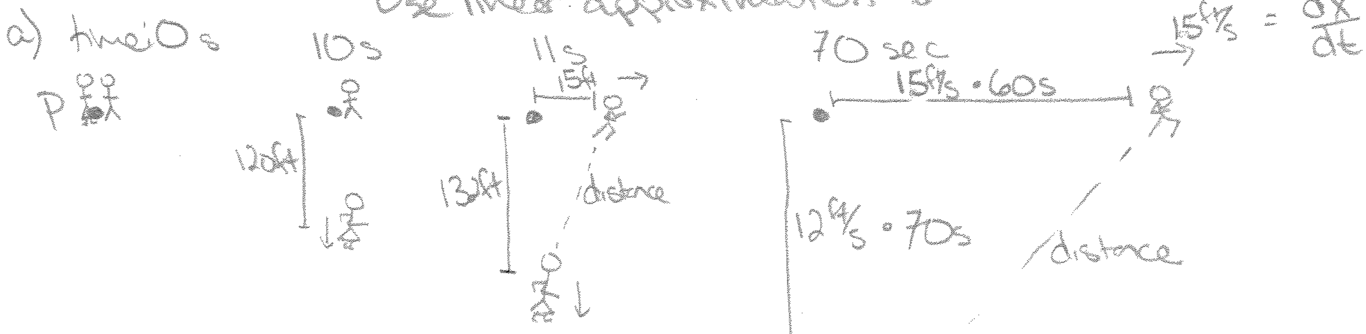
⇒ $y = \frac{1}{4}x + \frac{1}{4} + 1$ ⇒ $y = \frac{1}{4}x + \frac{5}{4}$

height = y value on dashed line when $x = +3$ so $\frac{1}{4}(+3) + \frac{5}{4} = \frac{8}{4} = 2$ is the height

10. [] (Story Problem Worksheet) Choose *ONE* of the following. Clearly identify which of the two you are answering and what work you want to be considered for credit. No, doing both questions will not earn you extra credit.

(a) (§3.9 #21) [5] Ryan and Stella were being chased by a pack of zombies. At point P they decided to split up and Stella ran south at 12 ft/s. Ryan waited for ten seconds to try to draw most of the zombies towards him and then started to run east at 15 ft/s. One minute later the two of them are still alive and running in their respective directions. At what rate are Ryan and Stella moving apart at this instant?

(b) If a current i passes through a resistor with resistance r, Ohm's Law states that the voltage drop is $v = ri$. Assume that voltage remains a constant 20 volts. An unreliable resistor claims a resistance of 10 ohms but may be off by up to 1.5 ohms. Use linear approximation to Approximate the ~~relative~~ error ~~in~~ when calculating i ~~(consult page 254 if you don't know the definition of relative error)~~.



$$\begin{pmatrix}\text{distance}\\\text{between}\\\text{them}\end{pmatrix}^2 = \begin{pmatrix}\text{distance}\\\text{Ryan runs}\end{pmatrix}^2 + \begin{pmatrix}\text{distance}\\\text{Stella runs}\end{pmatrix}^2$$

$d^2 = x^2 + y^2$ where x = distance Ryan runs at time t
y = distance Stella runs at time t
d = distance between Ryan and Stella

we want $\left.\frac{dd}{dt}\right|_{t=70}$

So $\frac{d}{dt}[d^2] = \frac{d}{dt}[x^2 + y^2]$

$$\Rightarrow \frac{\cancel{2d}\,\frac{dd}{dt}}{\cancel{2d}} = \frac{2x\frac{dx}{dt} + 2y\frac{dy}{dt}}{2d}$$

$$\Rightarrow \frac{dd}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{d}$$

when $t = 70$

$x = 15\,\text{ft/s} \cdot 60\,\text{s} = 900\,\text{ft}$

$y = 12\,\text{ft/s} \cdot 70\,\text{s} = 840\,\text{ft}$

Recall $d^2 = x^2 + y^2$

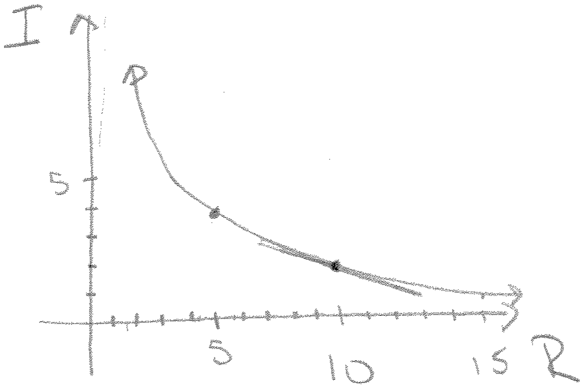
$\Rightarrow d = \sqrt{900^2 + 840^2}$

$$\text{So } \left.\frac{dd}{dt}\right|_{t=70} = \frac{900(15) + 840(12)}{\sqrt{900^2 + 840^2}} \approx 19.2\ \text{ft/s}$$

b) Ohm's law $V = RI$ we'll want to examine I as a function of R

$V = 20$ volts

$$\Rightarrow I = \frac{V}{R} = \frac{20}{R} = 20R^{-1}$$



to approximate the error we'll use a linear approximation of the function $I = 20/R$

ie the line tangent to $I = 20/R$ when $R = 10$.

equation of line:

$y = mx + b$ or $y - y_1 = m(x - x_1)$

where y is I and x is R

m = slope of line tangent to $I = 20/R$ when $R = 10$

$$= I'\Big|_{R=10}$$

$I' = 20(-1)R^{-2}$ by Power rule

$= -20R^{-2}$

$$\Rightarrow m = -20(10)^{-2} = \frac{-20}{10^2} = \frac{-2\cancel{0}}{10\cancel{0}} = \frac{-1}{5}$$

The line passes thru $(10, \frac{20}{10}) = (10, 2)$

so

$2 = -\frac{1}{5}(10) + b$ or $y - 2 = -\frac{1}{5}(x - 10)$

$\Rightarrow b = 2 + 2 = 4$ $\quad y - 2 = -\frac{1}{5}x + 2$

$I = -\frac{1}{5}R + 4$ $\quad I = -\frac{1}{5}R + 4$

If R is off by 1.5 ohms

R could be anywhere between

$10 - 1.5 = 8.5$ or $10 + 1.5 = 11.5$

$\Rightarrow$ the approximate I values would thus range from

$-\frac{1}{5}(8.5) + 4$ to $-\frac{1}{5}(11.5) + 4$

or from 2.3 to 1.7

If R was exact I would be $\frac{20}{10} = 2$

Thus an approx bound on the error is

$|2.3 - 2| = |1.7 - 2| = .3$ amps

Warning: the graph suggests that we are underestimating the error!