

Note: This is a practice exam and is intended only for study purposes. The actual exam will contain different questions and may have a different layout.

1. [] TRUE/FALSE: Let f and g be functions. Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F.

T (F) $\frac{3x+y}{3z} = \frac{x+y}{z}$

$$\frac{x+y}{z^2} = \frac{3(x+y)}{3z^2} \times \frac{3x+y}{3z}$$

T (F) $(x+y)^2 = x^2 + y^2$

$$(x+y)^2 = (x+y)(x+y) = x^2 + xy + xy + y^2 = x^2 + 2xy + y^2$$

T (F) $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ for all a only if $\lim_{x \rightarrow a} g(x) \neq 0$

T (F) If $\lim_{x \rightarrow \infty} f(x) = \infty$ and $\lim_{x \rightarrow \infty} g(x) = \infty$, then $\lim_{x \rightarrow \infty} [f(x) - g(x)] = 0$.
 note $f(x) = x$
 and $g(x) = x$

T (F) If f is continuous at a , then f is differentiable at a .

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$$\lim_{x \rightarrow \infty} (x-x^2) = \lim_{x \rightarrow \infty} x(1-x) =$$

$$= -\infty$$

Show your work for the following problems. The correct answer with no supporting work will receive NO credit (this includes multiple choice questions).

2. Find a formula for a function that satisfies all of the following criteria:

- has a vertical asymptote at $x = 2 \Rightarrow x-2$ factor in den
- a removable discontinuity at $x = 0$, and $\Rightarrow x-0$ factor in num and den
- horizontal asymptote at $y = -1 \Rightarrow \lim_{x \rightarrow \infty}$ fraction = -1

$$\times \frac{(x-0)}{(x-2)(x-0)} = \frac{x}{x^2-2x} \quad \text{note } \lim_{x \rightarrow \infty} \frac{x}{x^2-2x} = \lim_{x \rightarrow \infty} \frac{\frac{x}{x^2}}{1-\frac{2x}{x^2}} = 0$$

we need higher power of x in the numerator

$$\times \frac{x^2}{x^2-2x} \quad \text{note still v.a. @ } x=2 \text{ and } \lim_{x \rightarrow \infty} \frac{x^2}{x^2-2x} = \lim_{x \rightarrow \infty} \frac{\frac{x^2}{x^2}}{1-\frac{2x}{x^2}} = 1$$

and hole @ $x=0$
we're just off by a negative sign so

$$\frac{-x^2}{x^2-2x} \quad \text{works}$$

Note: there are many other solutions

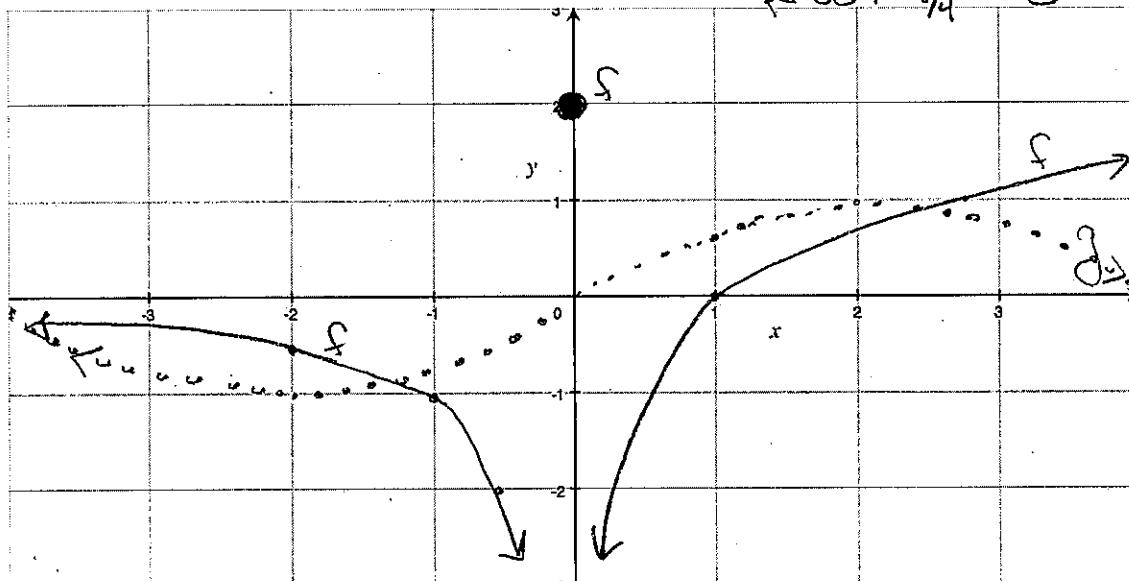
3. Given the rules of f and g below, graph both functions on the axis provided and evaluate the following

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x < 0, \\ 2 & \text{if } x = 0, \\ \ln x & \text{if } x > 0, \end{cases} \quad (\text{solid})$$

(dotted)

$$g(x) = \sin\left(\frac{\pi}{4}x\right)$$

$$\text{Period: } \frac{2\pi}{\frac{\pi}{4}} = 8$$



$$\lim_{x \rightarrow \infty} g(x)$$

doesn't exist.

$$\lim_{x \rightarrow 0} f(x)$$

$-\infty$

$$f(0)$$

2

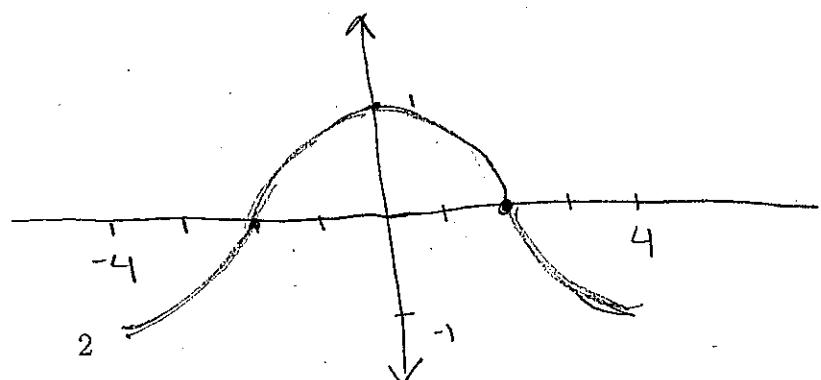
$$\lim_{x \rightarrow -2} (2f(x) + g(x))$$

$$= 2 \left[\lim_{x \rightarrow -2} f(x) \right] + \lim_{x \rightarrow -2} g(x) = 2\left(-\frac{1}{2}\right) + \sin\left(\frac{\pi}{4}(-2)\right) \\ = -1 + \sin\left(\frac{\pi}{2}\right) = -1 - 1 = -2$$

List any values that f is not continuous at:

not cont when $x=0$

O
b/c tangent line to g when
 $x=2$ is horizontal.
Graph $g'(x)$



4. [] Find the limit if it exists, or explain why it does not exist.

$$\lim_{x \rightarrow 3} \frac{x^2 - 4x + 3}{x^2 - 2x - 3} = \lim_{x \rightarrow 3} \frac{(x-3)(x-1)}{(x-3)(x+1)}$$

$$= \lim_{x \rightarrow 3} \frac{x-1}{x+1} = \frac{3-1}{3+1}$$

$$= \frac{2}{4} = \frac{1}{2}$$

$$\lim_{h \rightarrow 0} \frac{(1+h)^{-1} - 1}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{1+h} - 1}{h} = \lim_{h \rightarrow 0} \frac{\frac{1-(1+h)}{1+h}}{h} = \lim_{h \rightarrow 0} \frac{-h}{h(1+h)} = \lim_{h \rightarrow 0} \frac{-1}{1+h} = \lim_{h \rightarrow 0} \frac{-1}{1+0} = -1$$

$$= \lim_{h \rightarrow 0} \left[\frac{-h}{1+h} \div h \right] = \lim_{h \rightarrow 0} \frac{-h}{h(1+h)} = \lim_{h \rightarrow 0} \frac{-1}{1+h} = \lim_{h \rightarrow 0} \frac{-1}{1+0} = -1$$

$$= \lim_{h \rightarrow 0} \frac{-1}{1+h} = \frac{-1}{1+0} = -1$$

$$\lim_{x \rightarrow 1} \frac{2x-2}{|x-1|} = \lim_{x \rightarrow 1} \frac{2(x-1)}{|x-1|}$$

$$\text{if } x > 1$$

$$\lim_{x \rightarrow 1^+} \frac{2(x-1)}{|x-1|} = \lim_{x \rightarrow 1^+} \frac{2(x-1)}{x-1}$$

$$= \lim_{x \rightarrow 1^+} \frac{2}{x-1}$$

$$\lim_{x \rightarrow 1^+} 2 = 2$$

$$\text{if } x < 1$$

$$\lim_{x \rightarrow 1^-} \frac{2(x-1)}{|x-1|} = \lim_{x \rightarrow 1^-} \frac{2(x-1)}{-(x-1)}$$

$$= \lim_{x \rightarrow 1^-} \frac{2}{-(x-1)} = -2$$

Does not exist b/c left hand limit doesn't equal the right.

$$\lim_{x \rightarrow \infty} e^{-2x} \sin x$$

Note $-1 \leq \sin x \leq 1$

As b/c $e^{-2x} > 0$

$$-e^{-2x} \leq e^{-2x} \sin x \leq e^{-2x}$$

$$\text{Note } \lim_{x \rightarrow \infty} -e^{-2x} = 0$$

b/c graph transformations

$$\lim_{x \rightarrow \infty} e^{-2x} = 0 \text{ so}$$

$$\lim_{x \rightarrow \infty} \arctan(x^2 - x^4)$$

$$\lim_{x \rightarrow \infty} e^{-2x} \sin x = 0$$

$$\lim_{x \rightarrow \infty} \arctan(x^2(1-x^2))$$

$$= \arctan \lim_{x \rightarrow \infty} (x^2(1-x^2))$$

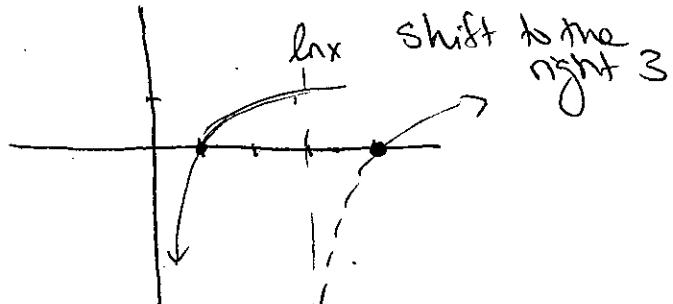
$$\text{and } \lim_{x \rightarrow \infty} x^2(1-x^2) = -\infty$$

By the graph of arctan then

$$\lim_{x \rightarrow \infty} \arctan(x^2 - x^4) = -\pi/2$$

$$\lim_{x \rightarrow \infty}$$

$$\lim_{x \rightarrow 3^+} \ln(x-3)$$



so $-\infty$.

5. Does $f(x) = 2x^3 + 6x^2 - 10x - 30$ have a root between 2 and 3? Explain your reasoning and cite theorems if you use any.

Notice $f(2) = 2 \cdot 2^3 + 6 \cdot 2^2 - 10 \cdot 2 - 30 = -10$

and $f(3) = 2 \cdot 3^3 + 6 \cdot 3^2 - 10 \cdot 3 - 30 = 48$.

The function f is a polynomial so ~~thus~~ f is cont.
This means to pass from (2, -10) to the point (3, 48),
the graph of f must pass through the x -axis & thus
have a root between 2 & 3. [Intermediate value Thm]

6. [] Find the equation for the line tangent to the graph of $y = \frac{1}{(x-2)^2}$, when $x = 3$.

looking for $y = mx + b$. let $g(x) = \frac{1}{(x-2)^2}$

$$m = \text{slope of the tangent} = g'(3) = \lim_{h \rightarrow 0} \frac{g(3+h) - g(3)}{h}$$

to $\frac{1}{(x-2)^2}$ at $x=3$

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{(3+h-2)^2} - \frac{1}{(3-2)^2}}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{(1+h)^2} - \frac{1}{1}}{h}$$

$$\lim_{h \rightarrow 0} \frac{1 - (1+h)^2}{(1+h)^2} \div h = \lim_{h \rightarrow 0} \frac{1 - 1 - 2h - h^2}{(1+h)^2} \cdot \frac{1}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-2h - h^2}{h(1+h)^2} = \lim_{h \rightarrow 0} \frac{h(-2-h)}{h(1+h)^2} = \frac{-2+0}{(1+0)^2} = -2$$

Notice the line passes through the point $(3, g(3))$

[b/c it touches the graph of g]

$$\text{or } (3, \frac{1}{(3-2)^2}) = (3, 1)$$

$$\text{so } 1 = -2(3) + b \Rightarrow 1 = -6 + b \Rightarrow b = 7$$

Thus $y = -2x + 7$ works.

7. Suppose that the motion of a ball can be described by the equation $f(t) = t^2 + t - 3$.
 Find the instantaneous velocity of the ball after 4 seconds.

velocity = $\frac{\Delta \text{position}}{\Delta \text{time}}$ so inst. velocity = $\lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h}$
 or rather
 $\lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \rightarrow 0} \frac{[(4+h)^2 + (4+h) - 3] - [4^2 + 4 - 3]}{h}$

$f'(t) = 2t + 1$ by § 3.1 so $f'(4) = 2 \cdot 4 + 1 = 8 + 1 = 9$ units/sec.

8. [] Using the definition, find the derivative of $f(x) = \sqrt{2x - \frac{1}{2}}$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\sqrt{2(x+h) - \frac{1}{2}} - \sqrt{2x - \frac{1}{2}}}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{2x+2h-\frac{1}{2}} - \sqrt{2x-\frac{1}{2}}}{h} \frac{(\sqrt{2x+2h-\frac{1}{2}} + \sqrt{2x-\frac{1}{2}})}{(\sqrt{2x+2h-\frac{1}{2}} + \sqrt{2x-\frac{1}{2}})} \\ &= \lim_{h \rightarrow 0} \frac{(2x+2h-\frac{1}{2}) - (2x-\frac{1}{2})}{h(\sqrt{2x+2h-\frac{1}{2}} + \sqrt{2x-\frac{1}{2}})} = \lim_{h \rightarrow 0} \frac{2h}{h(\sqrt{2x+2h-\frac{1}{2}} + \sqrt{2x-\frac{1}{2}})} = \lim_{h \rightarrow 0} \frac{2}{\sqrt{2x+2h-\frac{1}{2}} + \sqrt{2x-\frac{1}{2}}} \\ &= \frac{2}{\sqrt{2x+2 \cdot 0 - \frac{1}{2}} + \sqrt{2x-\frac{1}{2}}} = \frac{2}{\sqrt{2x-\frac{1}{2}} + \sqrt{2x-\frac{1}{2}}} = \frac{2}{2\sqrt{2x-\frac{1}{2}}} = \frac{1}{\sqrt{2x-\frac{1}{2}}} \end{aligned}$$

9. Describe 3 situations in which a function $f(x)$ could fail to be differentiable at a point.

if: f is not cont.

f has a kink

f has a vert.
tangent line

ex

