EXAM 1

TMath 124

Key

Practice

Note: This is a practice exam and is intended only for study purposes. The actual exam will contain different questions and may have a different layout.

1. [] TRUE/FALSE: Let f and g be functions. Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F.

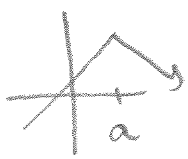
   T F $\frac{3x+y}{3z} = \frac{x+y}{z}$ $\qquad \frac{x+y}{z} = \frac{3(x+y)}{3z} = \frac{3x+3y}{3z}$

   T F $(x+y)^2 = x^2 + y^2$ $\qquad (x+y)^2 = (x+y)(x+y) = x^2+xy+xy+y^2$

   T F $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ for all a $\qquad$ only if $\lim_{x\to a} g(x) \neq 0$

   T F If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty}[f(x) - g(x)] = 0$.

   Let $f(x)=x$ and $g(x)=x^2$ then $\lim_{x\to\infty}(x-x^2) = \lim_{x\to\infty} x(1-x) = -\infty$

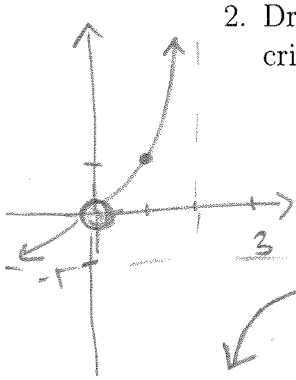
   T F If f is continuous at a, then f is differentiable at a.

   T F If f is differentiable at a, then f is continuous at a. Thm from class

Show your work for the following problems. The correct answer with no supporting work will receive NO credit (this includes multiple choice questions).

2. Draw a graph and then find a formula for a function that satisfies all of the following criteria: Note! there are many correct answers to this

   - has a vertical asymptote at $x = 2$ $\Rightarrow x-2$ is a factor in den
   - a removable discontinuity at $x = 0$, and $\Rightarrow x$ is factor in num + den
   - horizontal asymptote at $y = -1$. $\Rightarrow \lim_{x\to\infty}$ function $=-1$ or $\lim_{x\to-\infty}$ function $=-1$



$\cancel{\frac{x}{x(x-2)}}$ vert asy at $x=2$ ✓
removable disc at $x=0$ ✓
horiz asy ✗ b/c $\lim_{x\to\infty} \frac{x}{x(x-2)} = \lim_{x\to\infty} \frac{1}{x-2} = "\frac{1}{\text{Big}}"$ 0
the power on the den is getting too large..

$\cancel{\frac{x^2}{x(x-2)}}$ vert asy at $x=2$ ✓
removable disc at $x=0$ ✓
horiz asy ✗ b/c $\lim_{x\to\infty} \frac{x^2}{x(x-2)} = \lim_{x\to\infty} \frac{x}{x-2} = \lim_{x\to\infty} \frac{x/x}{\frac{x}{x} - \frac{2}{x}} =$
we're just off by a negative sign so $\frac{-x^2}{x(x-2)}$ works

x	$\frac{-x^2}{x(x-2)}$
0	DNE
1	-1
3	$\frac{-3}{+1}$

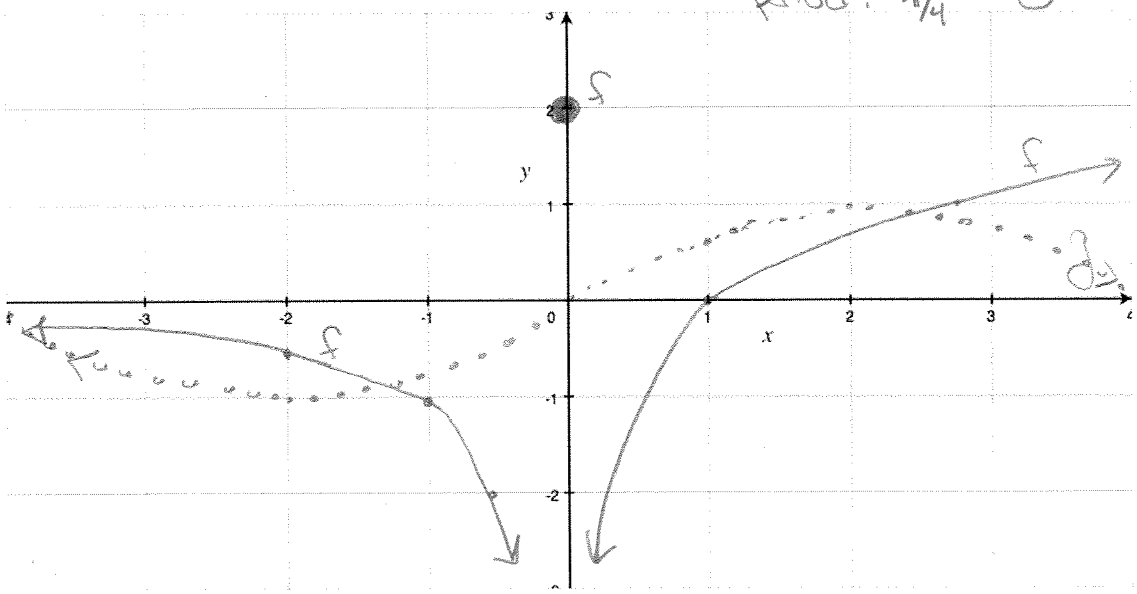
3. Given the rules of f and g below, graph both functions on the axis provided and evaluate the following

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x < 0, \\ 2 & \text{if } x = 0, \\ \ln x & \text{if } x > 0, \end{cases}$$ (solid)

(dotted)

$$g(x) = \sin(\tfrac{\pi}{4}x)$$

Period : $\frac{2\pi}{\pi/4} = 8$



$\lim_{x\to\infty} g(x)$

doesn't exist

$\lim_{x\to 0} f(x)$

$-\infty$

f(0)

2

$\lim_{x\to 1}[\pi f(x) \times g(x)]$

$= \pi \lim_{x\to 1} f(x) \cdot \lim_{x\to 1} g(x)$

$= \pi \cdot 0 \cdot \sin(\pi/4) = 0$

$\lim_{x\to -2}(2f(x) + g(x))$

$= 2\left[\lim_{x\to -2} f(x)\right] + \lim_{x\to -2} g(x) = 2\left(\frac{1}{2}\right) + \sin\left(\frac{\pi}{4}(2)\right)$

$= -1 + \sin\left(\frac{-\pi}{2}\right) = -1 - 1 = -2$

$g'(2)$

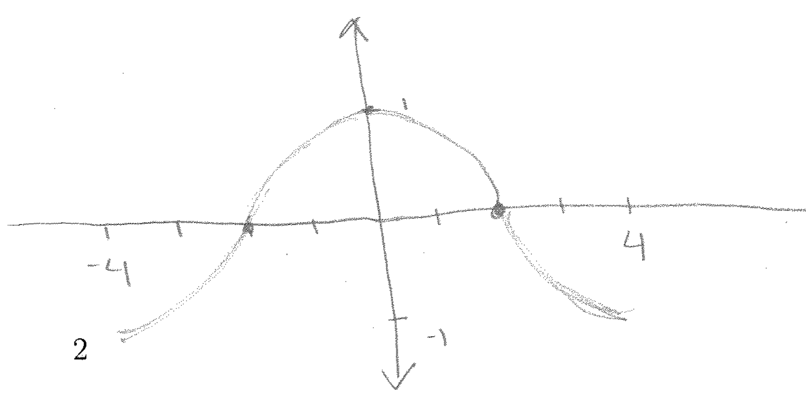
0

b/c tangent line to g when $x = 2$ is horizontal

List any values that f is *not* continuous at:

not cont when $x = 0$

Graph $g'(x)$

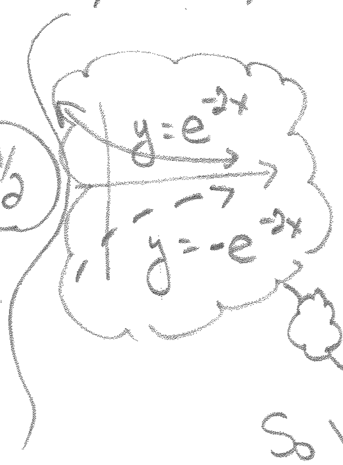


4. [] Find the limit if it exists, or explain why it does not exist.

$$\lim_{x\to 3}\frac{x^2-4x+3}{x^2-2x-3} = \lim_{x\to 3}\frac{\cancel{(x-3)}(x-1)}{\cancel{(x-3)}(x+1)}$$

$$= \lim_{x\to 3}\frac{x-1}{x+1}$$

$$= \frac{3-1}{3+1} = \frac{2}{4} = \frac{1}{2}$$



$$\lim_{x\to\infty} e^{-2x}\sin x$$

Recall $-1 \le \sin x \le 1$

Since e^{-2x} is greater than 0 for all x

$\Rightarrow \quad -1\cdot e^{-2x} \le e^{-2x}\sin x \le 1\cdot e^{-2x}$

$\Rightarrow \quad -e^{-2x} \le e^{-2x}\sin x \le e^{-2x}$

$\lim_{x\to\infty} -e^{-2x} = 0 = \lim_{x\to\infty} e^{-2x}$

So by the squeeze thm $\lim_{x\to\infty} e^{-2x}\sin x = 0$

$$\lim_{h\to 0}\frac{(1+h)^{-1}-1}{h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{1+h}-1}{h} = \lim_{h\to 0}\frac{\frac{1}{1+h}-\frac{1(1+h)}{1(1+h)}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{1-1-h}{(1+h)}}{h} = \lim_{h\to 0}\frac{-h}{1+h}\div\frac{h}{1}$$

$$= \lim_{h\to 0}\frac{-\cancel{h}}{1+h}\cdot\frac{1}{\cancel{h}} = \lim_{h\to 0}\frac{-1}{1+h}$$

$$= \frac{-1}{1+0} = -1$$

$$\lim_{x\to\infty}\arctan(x^2-x^4)$$

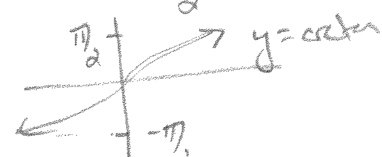
by continuity of arctan

$= \arctan\left(\lim_{x\to\infty}(x^2-x^4)\right)$

$= \arctan\left(\lim_{x\to\infty} x^2(1-x^2)\right)$

"Big $\cdot$ −Big = −Big"

$= \lim_{x\to -\infty}\arctan x = -\frac{\pi}{2}$



$$\lim_{x\to 0}\frac{3x^2}{3-\sqrt{9-x^2}}$$

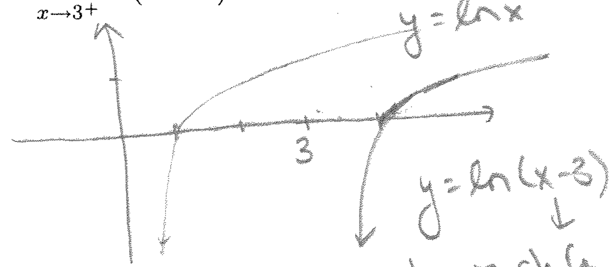
$$\lim_{x\to 0}\frac{3x^2}{3-\sqrt{9-x^2}} \qquad \cancel{\frac{3\cdot 0}{3-\sqrt{9-0}} = \frac{0}{0}}$$

$$\lim_{x\to 0}\frac{3x^2}{3-\sqrt{9-x^2}}\cdot\frac{(3+\sqrt{9-x^2})}{(3+\sqrt{9-x^2})} = \lim_{x\to 0}\frac{3x^2(3+\sqrt{9-x^2})}{9-(9-x^2)}$$

$$= \lim_{x\to 0}\frac{3x^2(3+\sqrt{9-x^2})}{x^2} = \lim_{x\to 0}\frac{3(3+\sqrt{9-x^2})}{1}$$

$$= 3(3+\sqrt{9-0^2}) = 3(3+\sqrt{9}) = 3\cdot 6 = 18$$

$$\lim_{x\to 3^+}\ln(x-3)$$



so $-\infty$
by examining the graph

5. Does $f(x) = 2x^3 + 6x^2 - 10x - 30$ have a root between 2 and 3? Explain your reasoning and cite theorems if you use any.

Notice $f(2) = 2\cdot 2^3 + 6\cdot 2^2 - 10\cdot 2 - 30 = -10$

and $f(3) = 2\cdot 3^3 + 6\cdot 3^2 - 10\cdot 3 - 30 = 48$.

The function f is a polynomial so ~~this~~ f is cont. This means to pass from $(2,-10)$ to the point $(3,48)$, the graph of f must pass through the x-axis & thus have a root between 2 & 3. [Intermediate value Thm]

6. [] Find the equation for the line tangent to the graph of $y = \frac{1}{(x-2)^2}$, when $x = 3$.

Looking for $y = mx + b$. Let $g(x) = \frac{1}{(x-2)^2}$

m = slope of line tangent to $\frac{1}{(x-2)^2}$ at $x=3$ $= g'(3) = \lim_{h\to 0} \frac{g(3+h)-g(3)}{h}$

$g'(3) =$

$$= \lim_{h\to 0} \frac{\frac{1}{(3+h-2)^2} - \frac{1}{(3-2)^2}}{h} = \lim_{h\to 0} \frac{\frac{1}{(1+h)^2} - \frac{1}{1}}{h}$$

$$= \lim_{h\to 0} \frac{1-(1+h)^2}{(1+h)^2} \div \frac{h}{1} = \lim_{h\to 0} \frac{1-1-2h-h^2}{(1+h)^2} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{-2h-h^2}{h(1+h)^2} = \lim_{h\to 0} \frac{\cancel{h}(-2-h)}{\cancel{h}(1+h)^2} = \frac{-2-0}{(1+0)^2} = -2$$

Notice the line passes through the point $(3, g(3))$

[b/c it touches the graph of g]

or $(3, \frac{1}{(3-2)^2}) = (3, 1)$

So $1 = -2(3) + b \rightsquigarrow 1 = -6 + b \rightsquigarrow b = 7$

Thus $y = -2x + 7$ works.

7. Suppose that the motion of a ball can be described by the equation $f(t) = t^2 + t - 3$. Find the instantaneous velocity of the ball after 4 seconds.

$$\text{velocity} = \frac{\Delta \text{ position}}{\Delta \text{ time}} \quad \text{so inst. velocity} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

or rather

$$\lim_{h \to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0} \frac{[(4+h)^2 + (4+h) - 3] - [4^2 + 4 - 3]}{h}$$ ~~blah~~ yuk...

$f'(t) = 2t + 1$ by §3.1 so $f'(4) = 2 \cdot 4 + 1 = 8 + 1 = 9 \ \frac{\text{units}}{\text{sec.}}$

8. [] Using the **definition**, find the derivative of $f(x) = \sqrt{2x - \frac{1}{2}}$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{2(x+h) - \frac{1}{2}} - \sqrt{2x - \frac{1}{2}}}{h} = \lim_{h \to 0} \frac{\sqrt{2x + 2h - \frac{1}{2}} - \sqrt{2x - \frac{1}{2}}}{h} \cdot \frac{(\sqrt{2x+2h-\frac{1}{2}} + \sqrt{2x - \frac{1}{2}})}{(\sqrt{2x+2h-\frac{1}{2}} + \sqrt{2x - \frac{1}{2}})}$$

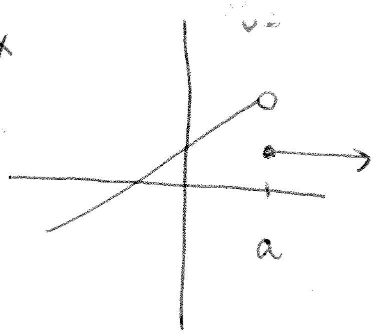
$$= \lim_{h \to 0} \frac{(2x + 2h - \frac{1}{2}) - (2x - \frac{1}{2})}{h[\sqrt{2x+2h-\frac{1}{2}} + \sqrt{2x - \frac{1}{2}}]} = \lim_{h \to 0} \frac{2\cancel{h}}{\cancel{h}[\sqrt{2x+2h-\frac{1}{2}} + \sqrt{2x - \frac{1}{2}}]} = \lim_{h \to 0} \frac{2}{\sqrt{2x+2h-\frac{1}{2}} + \sqrt{2x - \frac{1}{2}}}$$

$$\frac{2}{\sqrt{2x + 2 \cdot 0 - \frac{1}{2}} + \sqrt{2x - \frac{1}{2}}} = \frac{2}{\sqrt{2x - \frac{1}{2}} + \sqrt{2x - \frac{1}{2}}} = \frac{2}{2\sqrt{2x - \frac{1}{2}}} = \frac{1}{\sqrt{2x - \frac{1}{2}}}$$

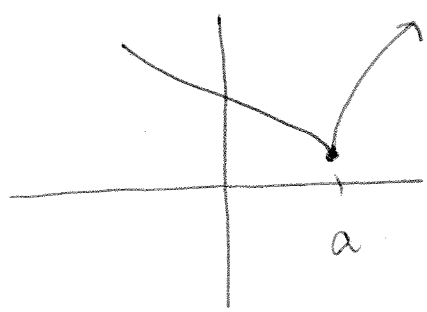
9. Describe 3 situations in which a function $f(x)$ could ***fail*** to be differentiable at a point.

if: f is not con't

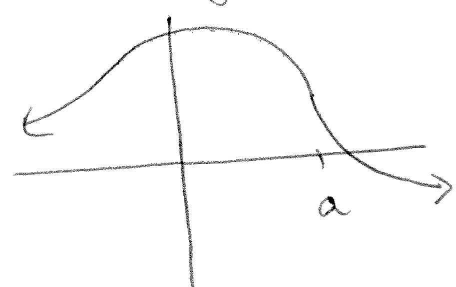
ex



f has a 'kink'



f has a vert. tangent line



10. [5] (Story Problem Worksheet) Choose *ONE* of the following. Clearly identify which of the two you are answering and what work you want to be considered for credit. No, doing both questions will not earn you extra credit.

   (a) The toll T charged for driving on the 520 bridge depend on the time of day. The Pay by Mail rates are given in the table below. Sketch a graph of T as a function of time t, measured in hours since past midnight. Then point out any discontinuities of this function and describe their significance to someone who uses the road.

Monday-Fridays	Pay By Mail	Mondays-Fridays	Pay By Mail
Midnight to 5am	0	2pm to 3pm	\$4.30
5am to 6am	\$3.10	3pm to 6pm	\$5.00
6am to 7am	\$4.30	6pm to 7pm	\$4.30
7am to 9am	\$5.00	7pm to 9pm	\$3.75
9am to 10am	\$4.30	9pm to 11pm	\$3.10
10am to 2pm	\$3.75	11pm to Midnight	0

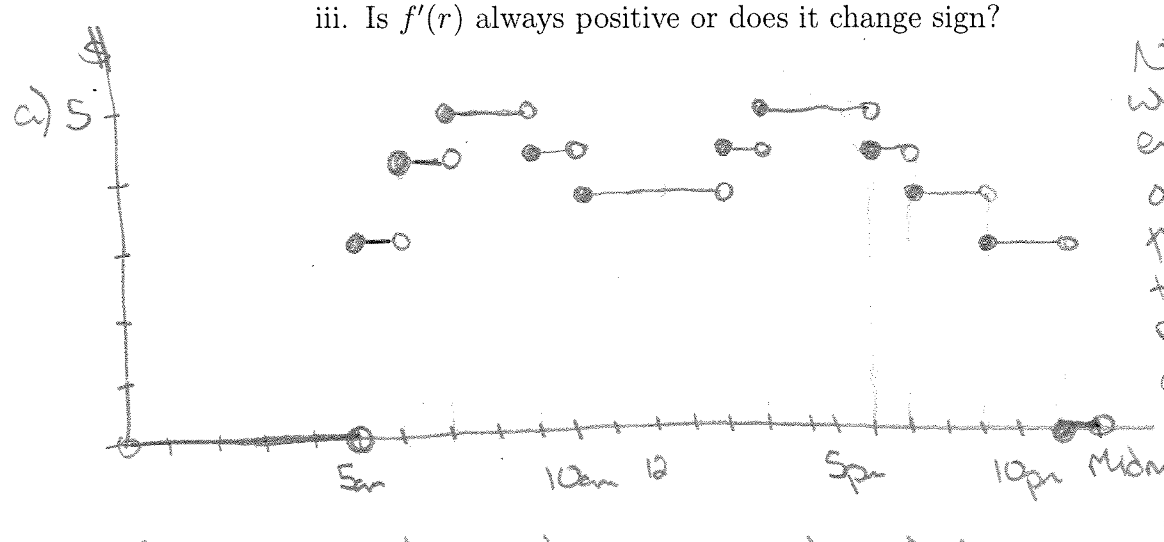
   (b) The total cost of repaying a student loan at an interest rate of $r\%$ per year is $C = f(r)$.

      i. What is the meaning of the derivative $f'(r)$?. What are its units?

      ii. What does the statement $f'(10) = 1200$ mean?

      iii. Is $f'(r)$ always positive or does it change sign?

a)



Note that it isn't clear what happens at exactly 7am (is it \$4.30 or \$5.00?). My guess is that the state would transition to the new rate but I'd accept another answer as long as you draw a function whose domain is Midnight to Mid

The graph is discontinuous anytime that the toll rate is changing from one rate to another, so at 5am, 6am, 7am, 10am, 2pm, 3pm, 6pm, 7pm, 9pm & 11p

b) i) $f'(x)$ is the $\dfrac{\text{change in cost of the total loan}}{\text{change in interest rate}}$ ⟹ units \$/%

ii) $f'(10) = 1200$ means increasing the interest rate from 10% to 11% would increase the total loan amount by about \$1200

iii) It's always positive since increasing the interest rate will always increase the total amount of the loan.