Key

Final Tuesday

TMath 124

Winter 2021

Show *all* your work (numerically, algebraically, or geometrically) for the following problems. Supporting work is needed to earn credit.

1. The work for the following problems is *wrong*. Explain why the solution is wrong *and* then find the correct solution (numerically, graphically, or algebraically).

   (a) [5] Find $\frac{dy}{dx}$ given $y = \log_2(x^2(x^4 + 2x^2))$.

   $$\frac{dy}{dx} = \frac{1}{(x^6+2x^4)\ln 2}(6x^5+8x^3)$$

   $$\begin{aligned} y &= \log_2((x^2)(x^4 + 2x^2)) \\ &= \log_2(x^6 + 2x^4) \\ \frac{dy}{dx} &= \frac{1}{x\ln(2)}(x^6 + 2x^4) + \log_2 \cdot (6x^5 + 8x^3) \end{aligned}$$

   ok

   -OR- use log properties

   not a product but composition.

   need to use the chain rule! $\log_2(x^6+2x^4)$

   (b) [4] Find $\frac{dy}{dx}$ given $\lim_{\theta \to \frac{\pi}{2}^-} \frac{\sin(2\theta)}{1 - \cos(2\theta)}$

   $$\begin{aligned} \lim_{\theta \to \frac{\pi}{2}^-} \frac{\sin(2\theta)}{1 - \cos(2\theta)} &\overset{L'H}{=} \lim_{\theta \to \frac{\pi}{2}^-} \frac{\cos 2\theta \cdot 2}{0 - (-\sin 2\theta \cdot 2)} \\ &= \lim_{\theta \to \frac{\pi}{2}^-} \frac{\cos 4\theta}{\sin 4\theta} \\ &= \frac{\cos 2\pi}{\lim_{\theta \to \frac{\pi}{2}^-} \sin 4\theta} = \frac{1}{0^-} = -\infty \end{aligned}$$

   not indeterminate

   note $\lim_{\theta \to \pi/2^-} \frac{\sin(2\theta)}{1-\cos(2\theta)} = \frac{0}{1-(-1)} = \frac{0}{2} = 0$

   (so we cannot use L'Hopital not of the form "$\frac{\infty}{\infty}$" or "$\frac{0}{0}$")

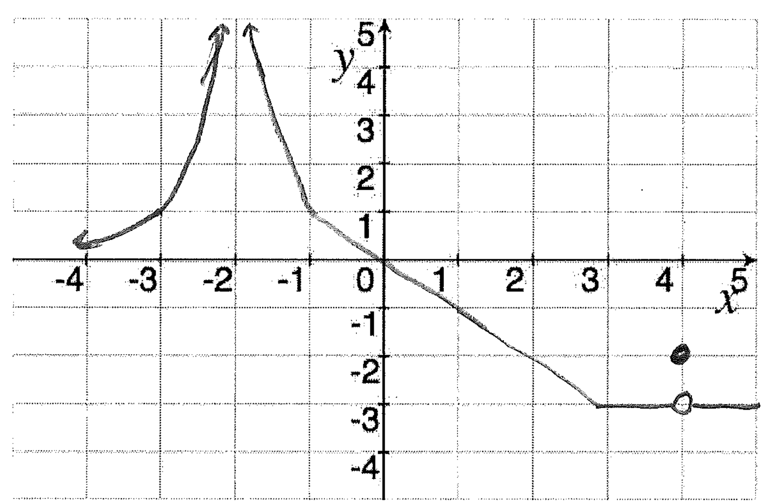
2. [5] Draw a graph for a function $\alpha(x)$, that satisfies *all* of the following:

   (a) α is continuous on the interval $(-2, 4)$,

   (b) $\lim_{x \to -2} \alpha(x) = \infty$,

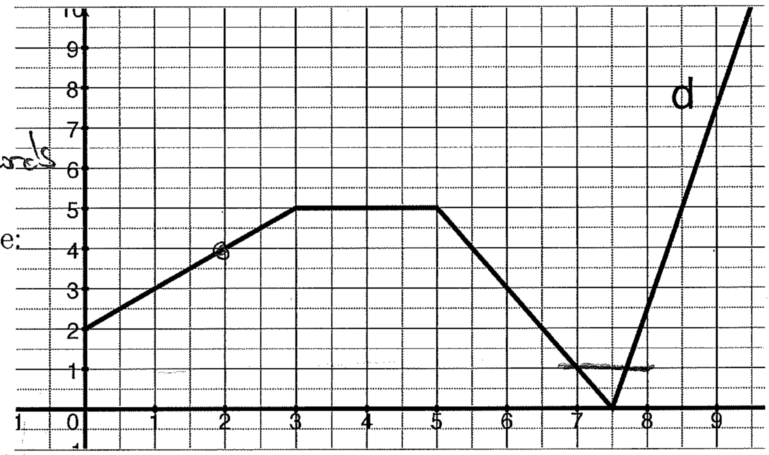
   (c) $\alpha'(2) = -1$, and

   (d) $\lim_{x \to 4} \alpha(x) \neq \alpha(4)$.



Note there are lots of answers

3. The following graph is a function, d, that returns the *distance* (in feet) a fly is from a spider web after t seconds.



(a) [3] Is the fly ever 1 foot away? If so, when?

yes $\approx t=7$ and 7.75 seconds

(b) [5] Estimate the following, if possible:

$$\lim_{t\to 6}\left(\frac{d(t)}{3}-2\right) = \frac{1}{3}\left(\lim_{t\to 6} d(t)\right)-2$$

$$= \frac{1}{3}\cdot 3 - 2 = 1-2$$

$$= -1$$

$d(2) = 4$

$\frac{d}{dt}d|_{t=2}$ = slope of the tangent line @ $t=2$

$= \frac{\text{rise}}{\text{run}} = \frac{1}{1}$

(c) [3] What is the speed of the fly when $t = 2$ and is the fly moving towards or away from the web?

from (b) we want $\left.\frac{dd}{dt}\right|_{t=2}$ which is 1 ft/sec and away from web

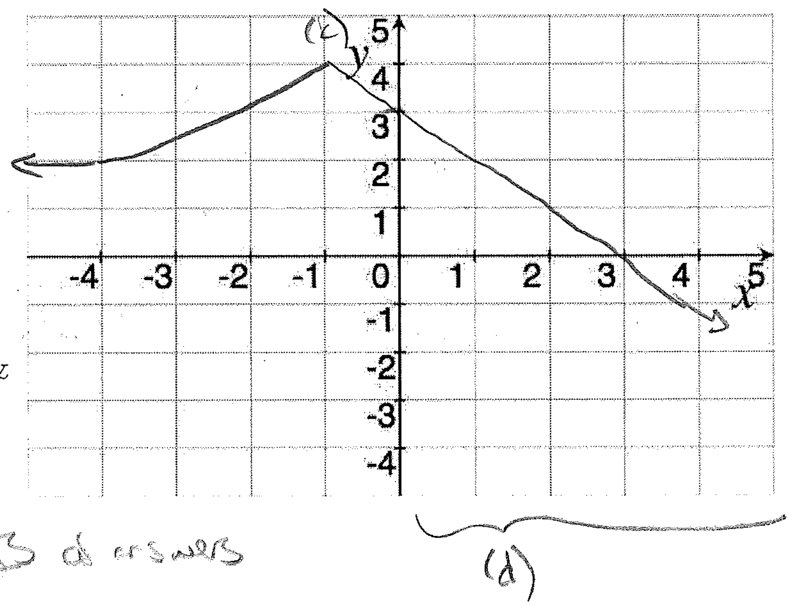
4. [5] Draw a graph for a function $\beta(x)$, that satisfies all of the following:

(a) $\lim_{x\to-\infty} \beta(x) = 2$,

(b) β is continuous on the interval $(-2, 3)$,

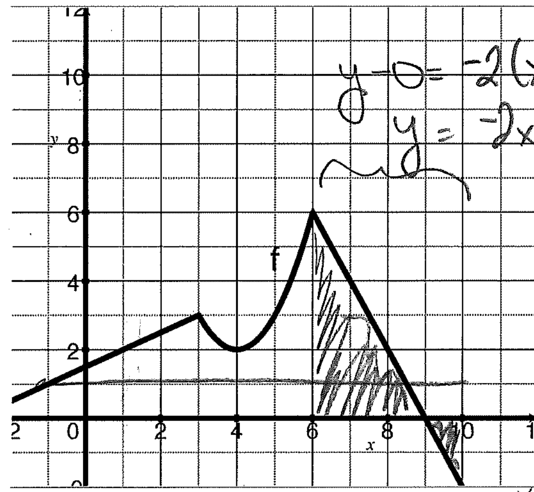
(c) $\beta'(-1)$ does not exist, &

(d) $\beta'(x) < 0$ when $0 < x$.



note, there are lots of answers

5. Let f be the function graphed on the right. Let $h(x) = 2^x$. Using this and the information about g given below, find the following (if possible!):



$y - 0 = -2(x-9)$

$y = -2x+18$

$g(2) = 1 \qquad g'(2) = -6$

$g(9) = 0 \qquad g'(9) = 5$

(a) [2] Estimate all x so that $f(x) = 1$.

$\approx x = -1$

(b) [2] Estimate all x so that $f'(x) = 0$.

i.e. horizontal tangent

$\approx x = 4$

(c) [3] Find $(f \cdot h)'(1)$.

Product rule

note $h(x) = 2^x$

$h'(x) = 2^x \ln 2$

$= f(1)h'(1) + f'(1)h(1)$

$= 2 \cdot 2^1(\ln 2) + \frac{1}{2} \cdot 2^1$

(d) [3] Find $\frac{d}{dx}(g(h(x)))|_{x=1}$.

Chain rule

$= g'(h(1)) \cdot h'(1) = g'(2) \cdot 2^1(\ln 2) = 1 \cdot 2^1(\ln 2) = 2(\ln 2)$

(e) [3] Find $\lim_{x \to 9} \frac{f(x)}{g(x)}$ LIMIT??

$= \frac{f(9)}{g(9)} = \text{"}\frac{0}{0}\text{"}$ indeterminant

$\overset{L'H}{=} \lim_{x \to 9} \frac{f'(x)}{g'(x)} = \frac{f'(9)}{g'(9)} = \frac{-2/1}{5} = \frac{-2}{5}$

(f) [2] Find the equation of the line tangent to g at $x = 9$.

Looking for $y = mx + b$ or $y - y_1 = m(x - x_1)$

m = slope of line tangent to g @ $x=9$
$= g'(9)$
$= 5$

thru $(9, g(9))$
or $(9, 0)$

so $y - 0 = 5(x-9)$

or

$y = 5x - 45$

(g) [2] Identify the area on the graph described by $\int_6^{10} f(x)\,dx$.

Shaded region except region below axis is considered negative

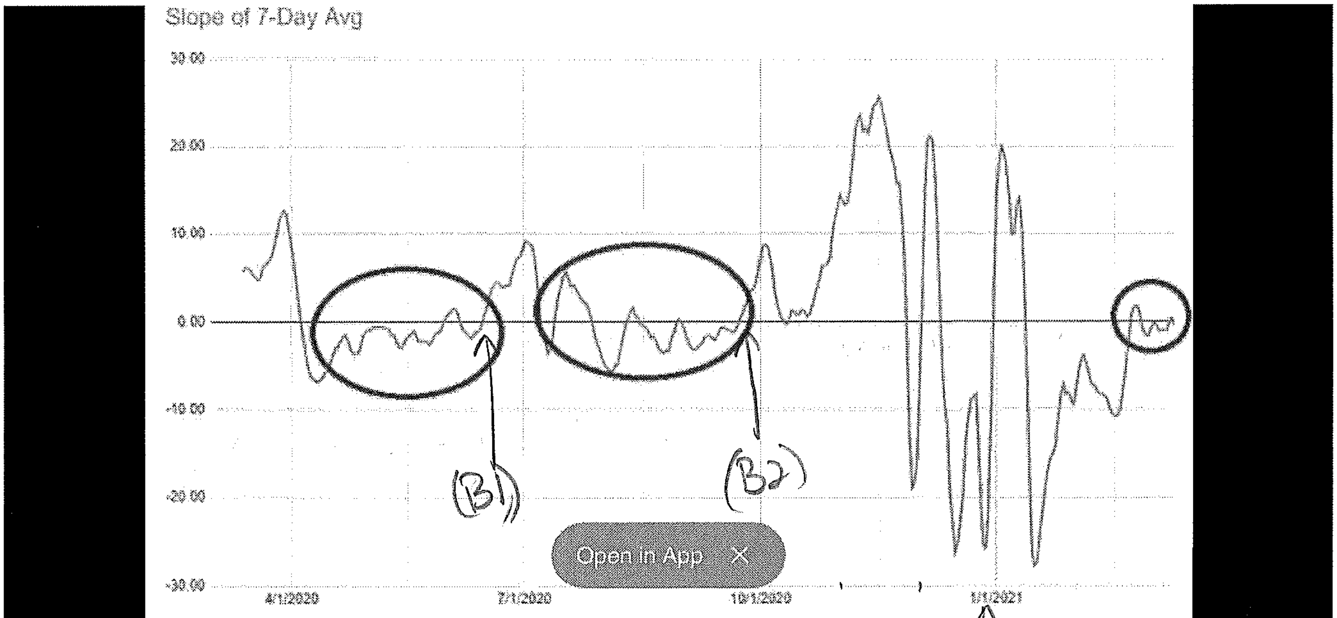
(h) [2] Find the value of $\int_6^{10} f(x)\,dx$.

$\frac{1}{2} \cdot 3 \cdot 6 - \frac{1}{2} \cdot 1 \cdot 2$

$9 - 1 = 8$

or $\int_6^{10} -2x + 18\,dx$

$= -x^2 + 18x\Big]_6^{10} = (-100 + 180) - (-36 + 18 \cdot 6)$
$= 8$



end of December

6. Use the graph provided by JCRooks on CoronavirusWA Reddit on Mar 13th duplicated above for the following questions. JCRooks is plotting the SLOPE of the 7-day average of new Covid-19 cases in King County over time.

   (a) [2] Describe what is happening to the 7-day average of new Covid-19 cases in King County at the end of December.

   The slope of the 7-day average is negative but moving towards a positive value. This means the 7-day average was dropping but is starting to increase (so likely we have a local minimum of cases)

   (b) [3] Identify a time that the 7-day average of new Covid-19 cases was at a local minimum. Explain/justify your answer.

   The transition from Dec to Jan 1st is likely a local min as described above in (a)
   B1 and B2 are also local minimums as the slope transitions from negative ↘ to positive ↗.

   (c) [2] Why do you think JCRooks provided this graph of the *slope* instead of the 7-Day average of new Covid-19 cases directly?

   Plotting the 7-day average of new cases of Covid-19 can be a bit harder to see if the spread (ie rate) is slowing down. The axis is a bit harder to process with 7-day average of new cases too (it gets pretty large).
   Also the rate is a better indicator of how quickly the illness is spreading.

7. [4] Find $\frac{dy}{dx}$ where $y = \frac{x^{0.5x}}{\arctan(x)}$

note x is in the exponent
⇒ cannot use power rule
note x is in the base
⇒ cannot use exponent rule
have to use logarithmic differentiation

$$\ln y = \ln\left(\frac{x^{.05x}}{\arctan(x)}\right)$$

$$\ln y = \ln(x^{.05x}) - \ln(\arctan(x))$$

$$\underset{(\text{chain})}{\ln y} = \underset{(\text{product})}{.05x\ln(x)} - \underset{(\text{chain})}{\ln(\arctan(x))}$$

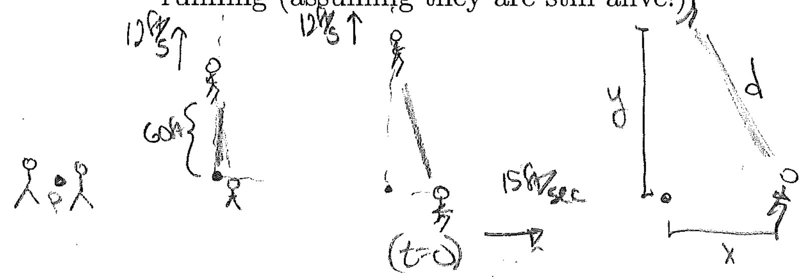
$$\frac{d}{dx} \Rightarrow \frac{1}{y}\frac{dy}{dx} = .05x\frac{1}{x} + .05\ln(x) - \frac{1}{\arctan(x)}\cdot\frac{1}{x^2+1}$$

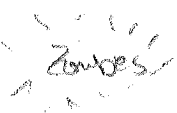
$$\Rightarrow \frac{dy}{dx} = y\left[.05 + .05(\ln(x))\right] - \frac{1}{(x^2+1)\arctan(x)}$$

8. Ryan and Stella were being chased by a pack of zombies. At point P they decided to split up and Ryan ran north at 12 ft/s. Stella waited for five seconds to try to draw most of the zombies towards her and then started to run east at 15 ft/s.

(a) [3] Find an equation relating the speed that Ryan and Stella are moving apart to other variables.

(b) [2] At what rate are Ryan and Stella moving apart two minutes after Stella started running (assuming they are still alive!)





a) $d^2 = x^2 + y^2$

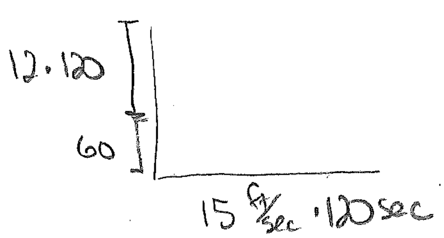
note $dx/dt = 15$ $\quad dy/dt = 12$

b) find $\left.\frac{dd}{dt}\right|_{t=120\text{ sec}}$

$$d^2 = x^2 + y^2$$

$$2d\frac{dd}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

When t=120 sec:



$$y = 60 + 12\cdot 120 = 60 + 1440 = 1500$$

$$x = 15\cdot 120 = 1800$$

and $d \Rightarrow d^2 = 1800^2 + 1500^2$

$$d = \sqrt{1800^2 + 1500^2}$$

so $2d\frac{dd}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

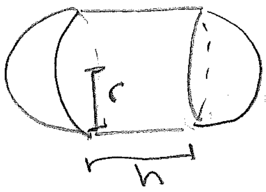
$$2\left(\sqrt{1800^2+1500^2}\right)\left.\frac{dd}{dt}\right|_{t=120} = 2(1800)(15) + 2(1500)(12)$$

(divide both sides by $2\left(\sqrt{1800^2+1500^2}\right)$)

9. An industrial tank is formed by adjoining two hemispheres to the ends of a right circular cylinder. The total volume of the tank is 5000 cubic feet. The hemispherical ends cost \$14 per square foot where as the sides cost only \$8 per square foot.

   (a) [4] Find a function that describes the cost of the tank dependent only on one variable.

   (b) [2] Outline the steps needed to find the dimensions to minimize cost ~~maximum enclosed area~~. Do *not* perform the steps!!!



$$\text{Volume} = \frac{4}{3}\pi r^3 + \pi r^2 \cdot h = 5000\text{ ft}^3 \quad (*)$$

Cost = Cost of 2 hemispheres + Cost of cylinder (assume no internal materials)

$$= (4\pi r^2)(14) + 2\pi r h \cdot 8$$

note this has 2 variables

Use * to solve one variable...

$$\frac{4}{3}\pi r^3 + \pi r^2 h = 5000$$

$$\Rightarrow \pi r^2 h = 5000 - \frac{4}{3}\pi r^3$$

$$\Rightarrow h = \left(\frac{5000 - \frac{4}{3}\pi r^3}{\pi r^2}\right)$$

(a) $\rightarrow = 4\pi r^2 \cdot 14 + 2\pi r\left(\dfrac{5000 - \frac{4}{3}\pi r^3}{\pi r^2}\right)8$

b) Find Cost'(r) so we can find the critical #'s (where Cost'(r)=0 or Cost'(r) DNE)
Use first derivative test on each critical # to determine if is a local/absolute minimum.

10. [4] Choose a problem from this exam that you've already answered,

   (a) show a second way of approaching/building a solution, and

   (b) explain why you did not choose this second method initially.

#8) we can write a formula for x and y as functions of t.

$$y = 12t + 60 \quad \text{and} \quad x = 15t$$

so $d^2 = x^2 + y^2$

$$\Rightarrow d^2 = (15t)^2 + (12t+60)^2 = 15^2t^2 + 144t^2 + 24\cdot 60t + 60^2$$

we could take the derivative with respect to t directly

$$2d\,\frac{dd}{dt} = 15^2 \cdot 2t + 144 \cdot 2t + 24 \cdot 60$$

Then we need only substitute $t=120$, $d = \sqrt{(15\cdot 120)^2 + (12\cdot 120 + 60)^2}$ to get the answer

b) I dont have a calculator on me and there is alot of arithmetic this way...