Exam 1 **TMath 120** Practice

Key

NAME: This is a sample exam to be used for practice. This is *not* a template for the exam that will be given in class. Many of the questions on the exam will look quite different than those appearing here.

1. TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F. Let f be a function, and x, y, and z be real numbers with $z \neq 0$.

   T (F) $\frac{3x+y}{3z} = \frac{x+y}{z}$ $\quad \frac{x+y}{z} = \frac{3(x+y)}{3z} = \frac{3x+3y}{3z}$

   T (F) $(x+y)^2 = x^2 + y^2$ $\quad (x+y)^2 = (x+y)(x+y) = x^2 + xy + yx + y^2 = x^2 + 2xy + y^2$

   T (F) $|x| = x$ $\quad$ let $x = -2$ then $|-2| = 2 \neq -2$

   (T) F $\frac{3+5i}{1-2i} = -\frac{7}{5} + \frac{11}{5}i$ $\quad \frac{3+5i}{1-2i} \cdot \frac{1+2i}{1+2i} = \frac{3+6i+5i+10i^2}{1+2i-2i-4i^2} = \frac{3+11i-10}{1-4(-1)}$

   (T) F A cubic polynomial always has three complex roots.

Show your work for the following problems. The correct answer with no supporting work will receive NO credit (this includes multiple choice questions).

2. Find any real or imaginary x such that $3(7+x)^2 + 4 = 2$.

$-4 \quad -4$

$\frac{3(7+x)^2}{3} = \frac{-2}{3}$

$(7+x)^2 = -2/3$

$7 + x = \pm\sqrt{-2/3}$

$x = -7 \pm \sqrt{-2/3}$

$= -7 \pm i\sqrt{2/3}$

3. Find any real or imaginary x such that $\frac{1}{x+1} + \frac{1}{2} = \frac{1}{x+3}$.

$\frac{1}{x+1} + \frac{1}{2} = \frac{1}{x+3}$

$\frac{2}{2(x+1)} + \frac{x+1}{2(x+1)} = \frac{1}{x+3}$

$\frac{2+x+1}{2(x+1)} = \frac{1}{x+3}$

$\frac{x+3}{2(x+1)} = \frac{1}{x+3}$ $\quad \times(x+3)$

$\frac{(x+3)(x+3)}{2(x+1)} = 1$ $\quad 2(x+1)$

$x^2 + 6x + 9 = 2x + 2$

$x^2 + 4x + 7 = 0$

$x^2 + 4x + 4 + 7 = 4$

$(x+2)^2 + 7 = 4$

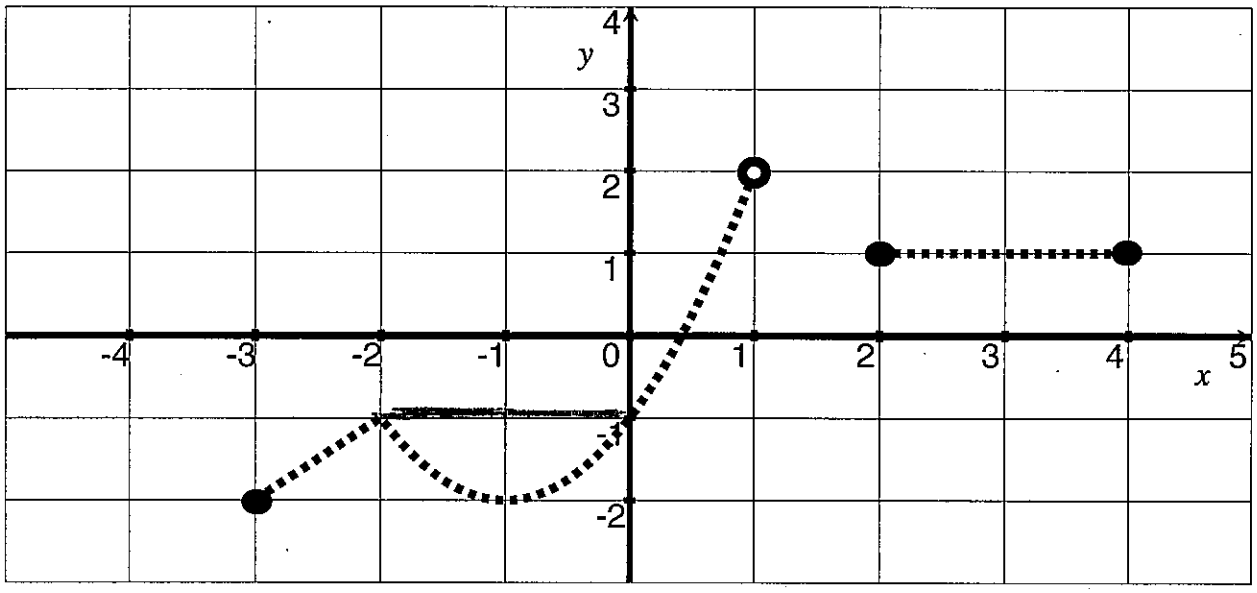
$-7 \quad -7$

$(x+2)^2 = -3$

$x + 2 = \pm\sqrt{-3}$

$x = -2 \pm i\sqrt{3}$

4. [4] Let f be the function comprised of two lines and a parabola that has only been shifted (not vertically stretched) and whose graph is below:



Estimate the following *if* possible:

$f(-3) = -2$

$\dfrac{f(-3)-1}{f(-1)} = \dfrac{-2-1}{-2}$

$= \dfrac{-3}{-2} = 3/2$

$f(1)$

is not defined

$(f \circ f)(0) = f(f(0))$

$= f(-1)$

$= -2$

$f(-1)f(2)$

$-2 \cdot 1 = -2$

$f(-1-2) = f(-3) = -2$

The average rate of change of f from $x = -2$ to $x = 0$

$\dfrac{f(-2)-f(0)}{-2-0} = \dfrac{(-1)-(-1)}{-2}$

$= \dfrac{-1+1}{-2} = 0$

i.e. slope of line drawn above

The piece-wise defined rule of f:

$$f(x) = \begin{cases} x+1 & \text{if } -3 \le x < -2 \\ (x+1)^2 - 2 & \text{if } -2 \le x < 1 \\ 1 & \text{if } 2 \le x \le 4 \end{cases}$$

5. Let $\alpha(x) = \frac{1}{x-2}$ and $\beta(x) = \frac{\sqrt{x+4}}{x}$.

(a) Find the domain of β.

all x but when den $=0$ or when shift under sqrt <0

so all all x but $x=0$ or when $x+4<0$

$\Rightarrow x<-4$

So $x \neq 0$ and $x \geq -4$ or $(-4,0) \cup (0,\infty)$

(b) Find the rule of $\beta \circ \alpha$. Simplify.

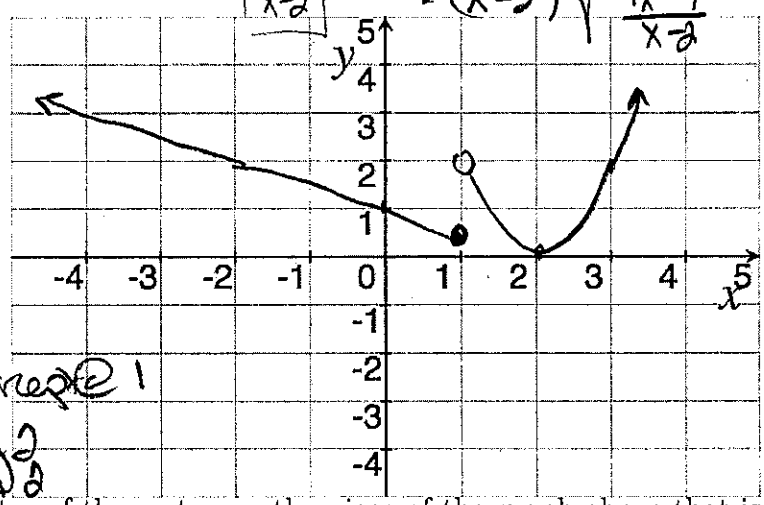
$(\beta \circ \alpha)(x) = \beta(\alpha(x))$

$= \beta\left(\frac{1}{x-2}\right) = \frac{\sqrt{\frac{1}{x-2}+4}}{\frac{1}{x-2}} = \sqrt{\frac{1}{x-2} + \frac{4}{1}\frac{x-2}{x-2}} \div \frac{1}{x-2}$

$= (x-2)\sqrt{\frac{4x-7}{x-2}}$

6. Let h be the *function* defined by:

$$h(x) = \begin{cases} -\frac{1}{2}x + 1 & x \leq 1 \\ 2(x-2)^2 & 1 < x \end{cases}$$



(a) Graph h. (Explaining graph transformations is worth partial credit.)

line w/ slope $-1/2$ y-intercept 1

parabola stretch vert by 2 + shifted right 2

(b) What are the coordinates of the vertex on the piece of the graph above that is a parabola?

$(2,0)$

(c) Identify the x-intercept(s).

2

(d) Find all possible input(s) so that $h(x) = 1$.

when $x=0$ and when $\Rightarrow 1/2 = (x-2)^2$

$1 = 2(x-2)^2$

$\pm\sqrt{1/2} = x-2$

$x = 2 \pm \frac{1}{\sqrt{2}}$

(e) What is the range h?

$[0,\infty)$

(f) On what interval(s) is h increasing?

$(2,\infty)$

7. [4] Given that $j(x) = -3x^2 + 6x - 2$. Write j in vertex (standard) form.

'bury the 3'

$\frac{y}{-3} = \frac{-3x^2 + 6x - 2}{-3}$

$-\frac{1}{3}y = x^2 - 2x + \frac{2}{3}$

add $\left(\frac{-2}{2}\right)^2$ to both sides

$+\left(\frac{-2}{2}\right)^2 \qquad +\left(\frac{-2}{2}\right)^2$

$\left(-\frac{1}{3}\right)y + 1 = x^2 - 2x + 1 + \frac{2}{3}$

factor $\left(-\frac{1}{3}\right)y + 1 = (x-1)^2 + \frac{2}{3}$

$-1 \qquad -1$

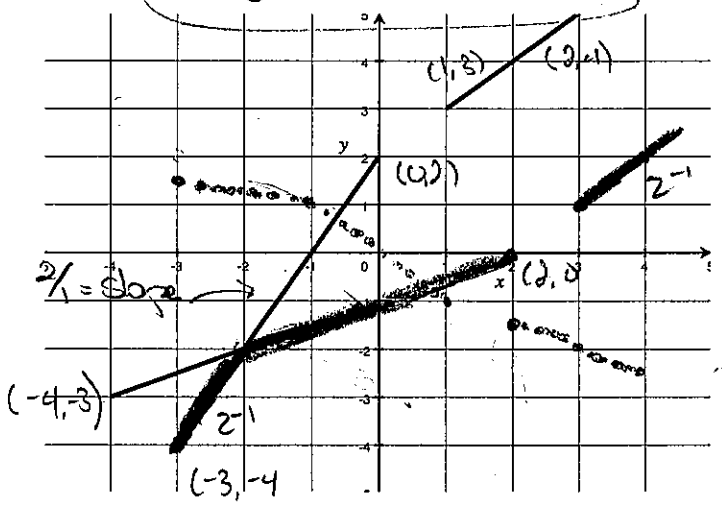
solve for y

$\left(-\frac{1}{3}\right)y = (x-1)^2 - \frac{1}{3}$

$y = -3\left[(x-1)^2 - \frac{1}{3}\right]$

$y = -3(x-1)^2 + 1$

8. Let z be the function whose graph is shown to the right.



(a) Find the equation of the line that passes though $(-3, 2)$ and makes a right angle when intersecting z.

Slope is $-\frac{1}{2}$ thru $(-3,2)$

$y - 2 = -\frac{1}{2}(x+3)$

(b) Draw the graph of z^{-1} *if* it exists.

-thick shaded graph

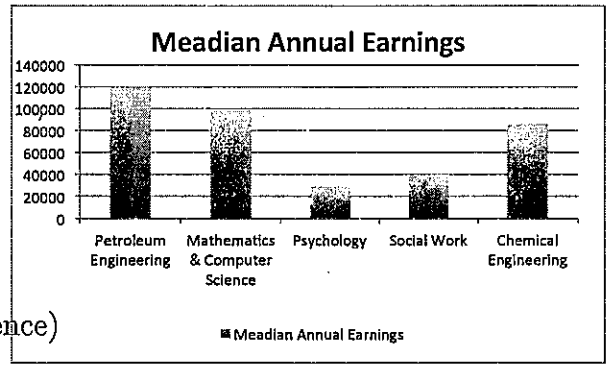
(c) Draw the graph of $-\frac{1}{2}z(x-1)$.

vertical stretch by $-\frac{1}{2}$ or mult y-coord by $-\frac{1}{2}$

then horz shift right by one unit

- dotted graph -

9. Let the domain of f be undergraduate majors and $f(x)$ be the median annual earnings of people with the the undergraduate major x.



(a) Is f a function? Why are why not?

Yep - each undergraduate will correspond to only one median annual earning

(b) Some data of f is shown in the graph on the right, what is f(Mathematics & Computer Science) and what does it mean?

f(Mathematics + Computer Science) ≈ 98,000

The median annual earnings of people with an undergraduate major in Math & CS is about 98,000.

(c) Find an x such that $f(x) > 100,000$.

Petroleum Engineering

10. Let $p(x) = \dfrac{x-5}{7x+5} + 3$.

(a) Given that p is one-to-one (ie has an inverse), find p^{-1}.

$$x = \frac{p^{-1}(x) - 5}{7p^{-1}(x) + 5} + 3$$

or

$$x = \frac{y-5}{7y+5} + 3$$

$$-3 \qquad -3$$

$$x - 3 = \frac{y-5}{7y+5}$$

$$(7y+5)(x-3) = \frac{y-5}{\cancel{7y+5}}\cancel{(7y+5)}$$

$$7yx - 21y + 5x - 15 = y - 5$$

$$-5x + 15 \qquad 5x + 15$$

$$7yx - 21y = y + 10 - 5x$$

$$-y \qquad -y$$

$$7yx - 22y = 10 - 5x$$

$$y(7x - 22) = 10 - 5x$$

$$y = \frac{10 - 5x}{7x - 22}$$

(b) Write the expression $p(a+h)$ and simplify.

$$p(\boxed{a+h}) = \frac{\boxed{a+h} - 5}{7\boxed{a+h} + 5} + 3$$

$$= \frac{a+h-5}{7a+7h+5} + 3$$

(c) Write the expression $\dfrac{p(a+h) - p(a)}{h}$ and simplify.

$$\frac{[p(\boxed{a+h}) - p(a)]}{h}$$

$$= \frac{\left[\frac{\boxed{a+h} - 5}{7\boxed{a+h} + 5} + 3\right] - \left[\frac{a-5}{7a+5} + 3\right]}{h}$$

$$= \left[\frac{a+h-5}{7(a+h)+5} + \cancel{3} - \frac{a-5}{7a+5} - \cancel{3}\right] \div h$$

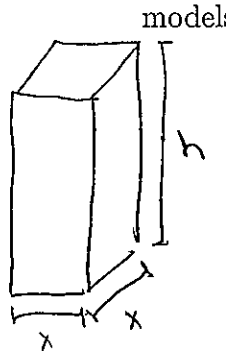
$$\left[\frac{7a+5}{7a+5}\,\frac{a+h-5}{7a+7h+5} - \frac{a-5}{7a+5}\,\frac{7a+7h+5}{7a+7h+5}\right] \div \frac{h}{1}$$

$$\frac{(7a+5)(a+h-5) - (a-5)(7a+7h+5)}{(7a+5)(7a+7h+5)} \cdot \frac{1}{h}$$

$$\frac{\cancel{7a^2} + \cancel{7ah} - \cancel{35a} + \cancel{5h} + 5h - \cancel{25} - \cancel{7a^2} - \cancel{7ah} - \cancel{5a} + \cancel{35a} + 35h + \cancel{25}}{(7a+5)(7a+7h+5)h}$$

$$\frac{5h + 35h}{(7a+5)(7a+7h+5)h} = \frac{40\cancel{h}}{(7a+5)(7a+7h+5)\cancel{h}}$$

11. A rectangular box with a volume of 60 ft^3 has a square base. Find a function that models its surface area S in terms of the length x of one side of its base.



Volume $= x \cdot x \cdot h$

$60\text{ft}^3 = x^2 h$

we want

Surface as a function of x

Surface = top + bottom + 4 sides

$= x^2 + x^2 + 4 \cdot xh$

$= \underbrace{2x^2 + 4xh}_{\text{right now S is a function of x and h}}$

we need to substitute h in for something else...

note: $60 = x^2 h$

So $h = \dfrac{60}{x^2}$

Then

Surface $= 2x^2 + 4xh$

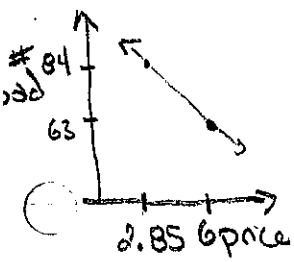
$= 2x^2 + 4x\left(\dfrac{60}{x^2}\right)$

$= 2x^2 + \dfrac{240}{x}$

12. [5] Choose *ONE* of the following. Clearly identify which of the two you are answering and what work you want to be considered for credit.
No, doing both questions will not earn you extra credit.

(a) You would like to set the price for a UWT fund-raising raffle. You did a similar thing last year and when you set the price to \$6 about 63 people bought tickets. The stats class did some research for you and reported that if ticket prices reduced by \$3.15, sales would increase by about 21 tickets. What price should you set the tickets so as to maximize income from ticket sales (to the nearest penny)?

(b) A manufacturer of soft drinks advertises their orange soda as "naturally flavored", although it contains only 5% orange juice. A new federal regulation stipulates that to be called "natural" a drink must contain at least 10% fruit juice. The manufacturer mixes their juices in closed 900 gallon containers (to avoid contamination). How much juice must they remove from the 900 gallon container and replace with pure orange juice to conform to the new regulation?



We'd like to maximize income = price · #sold

let p = price & q = #sold

So income = $p \cdot q$

Notice we have a linear relationship between p and q.

$$\text{slope} = \frac{-21}{3.15}$$

passes thru $(6, 63)$ so

$$q - 63 = \frac{-21}{3.15}(p-6)$$

or $q = -6.67p + 103$

So Income = $p \cdot q$ or

$= p(-6.67p + 103)$

$= -6.67p^2 + 103p$

which is a parabola opening down

$\Rightarrow$ the max is @ the vertex?

Income $= -6.67p^2 + 103p$

$-\frac{1}{6.67}$ Income $= p^2 + 15.44p$

$+7.72^2 \qquad +7.72^2$

$-\frac{1}{6.67}$ Income $+ 59.61 = (p + 7.72)^2$

So set the price to \$7.72

let x = the amount of pure orange to add

y = the amount of juice to keep

Note x also = amount of juice to remove

(1) $x + y = 900$ (total juice volume)

(2) $x + .05y = .1 \cdot 900$ (concentrations)

x ↑ juice add; y ↑ juice staying; $.1 \cdot 900$ ↑ juice level we want

From (1) $y = 900 - x$ so sub into (2)

$x + .05(900 - x) = .1 \cdot 900$

now we just solve for x.

$x + 45 - .05x = 90$

$.95x + \cancel{45} = 90$

$\cancel{-45} \quad -45$

$.95x = 45$

$\Rightarrow x = 47.37$ gal