

Practice

TMath 120

Final

NAME: This is a sample final to be used for practice. This is *not a template* for the Final that will be given in class. Many of the questions on the Final will look quite different than those appearing here.

[10] Let f & g , be functions.

T (F) $(f \circ g)(x) = (g \circ f)(x)$ ex $f(x) = x+1$ $g(x) = 3x$

T (F) $(\frac{f}{g})(x) = (\frac{g}{f})(x)$ $(fg)(x) = 3x+1 \neq 3(x+1) = (g \circ f)(x)$

T (F) $\sqrt{x^2} = x$ for all real numbers x . Let $x = -2$

(T) F If 2 is a root of g , then $g(2) = 0$. by definition of root

(T) F $\ln \frac{x}{y} = \ln x - \ln y$ for all positive numbers x and y .

(T) F $\log(\log(10)) = 0$. $\log 10 = 1 \Rightarrow \log(\log(10)) = \log(1) = 0$

T (F) $\sin^{-1}(\sin x) = x$ for all real numbers x . Let $x = 3\pi/2$

T (F) $\sin(\frac{\pi}{3} + x) = \sin \frac{\pi}{3} + x$ DEMAS $\Rightarrow \sin^{\pi/3+x} = (\sin \pi/3) + x$

(T) F If $\sin \theta \neq 0$ and $\tan \theta < 0$, then $\cos \theta < 0$

T (F) The range of \sin^{-1} is $[0, \pi]$

Right answers will *not* get credit without supporting work. Note "undefined" and "no solution" are possible answers.

1. Find all x such that

$$2(5 - (8 - x)^2)^{-\frac{1}{2}} - 1 = 0$$

$$\begin{aligned} 2(5 - (8 - x)^2)^{-\frac{1}{2}} - 1 &= 0 \\ \sqrt{5 - (8 - x)^2}^{-1} &= 1 \\ \sqrt{5 - (8 - x)^2} &= 1 \\ 2 = \sqrt{5 - (8 - x)^2} & \\ 4 = 5 - (8 - x)^2 & \\ 4 = 5 - (3 - x)^2 & \Rightarrow x = 7 \quad \text{or} \quad x = 9 \end{aligned}$$

2. [2] Explain what a function is.

A function consists of 2 sets (domain & range) and a rule between them such that every number in the domain is sent to exactly one output.

3. Given $m(x) = \frac{x}{x-5}$, and $n(x) = \sqrt{4x-8}$,

(a) The function m passes the horizontal line test. Find m^{-1} .

$$m(m^{-1}(x)) = x \quad \text{or} \quad x = \frac{y}{y-5} \quad \Rightarrow \quad xy - y = 5x \\ \Rightarrow x = \frac{m^{-1}(x)}{m^{-1}(x)-5} \quad \Rightarrow x(y-5) = y \quad \left\{ \begin{array}{l} y(x-1) = 5x \\ y = \frac{5x}{x-1} \end{array} \right. \\ \Rightarrow xy - 5x = y$$

(b) [4] If $p(x) = 3m(x+1)$, find the domain and rule of p .

$$p(x) = 3m(x+1) \quad \text{Domain: all } x \text{ so that denominator } \neq 0 \\ = 3 \frac{|x+1|}{|x+1|-5} \quad x+1 \neq 5 \\ = 3 \left(\frac{x+1}{x-4} \right) \quad x \neq 4 \\ \text{so } (-\infty, 4) \text{ and } (4, \infty)$$

(c) [3] Find the domain and rule of $n \circ m$.

$$n \circ m(x) = n \left(\frac{x}{x-5} \right) \quad \text{Domain: all } x \text{ so that denominator } \neq 0 \text{ and } \sqrt{4x-8} \geq 0 \\ = \sqrt{4 \left| \frac{x}{x-5} \right|} - 8 \quad x-5 \neq 0 \quad \frac{4x}{x-5} - 8 \geq 0 \\ \quad \quad \quad x \neq 5 \quad \quad \quad \frac{4x}{x-5} \geq 8$$

(d) [5] Find the domain and rule of $\frac{n}{m}$.

$$\left(\frac{n}{m} \right)(x) = \frac{n(x)}{m(x)} \quad \text{so } (-\infty, 0) \cup (0, 5) \cup (5, \infty) \\ = \frac{\sqrt{4x-8}}{\left(\frac{x}{x-5} \right)}$$

domain: all #s so that

den of den $\neq 0$ and dent ≥ 0

$$x-5 \neq 0$$

$$x \neq 5$$

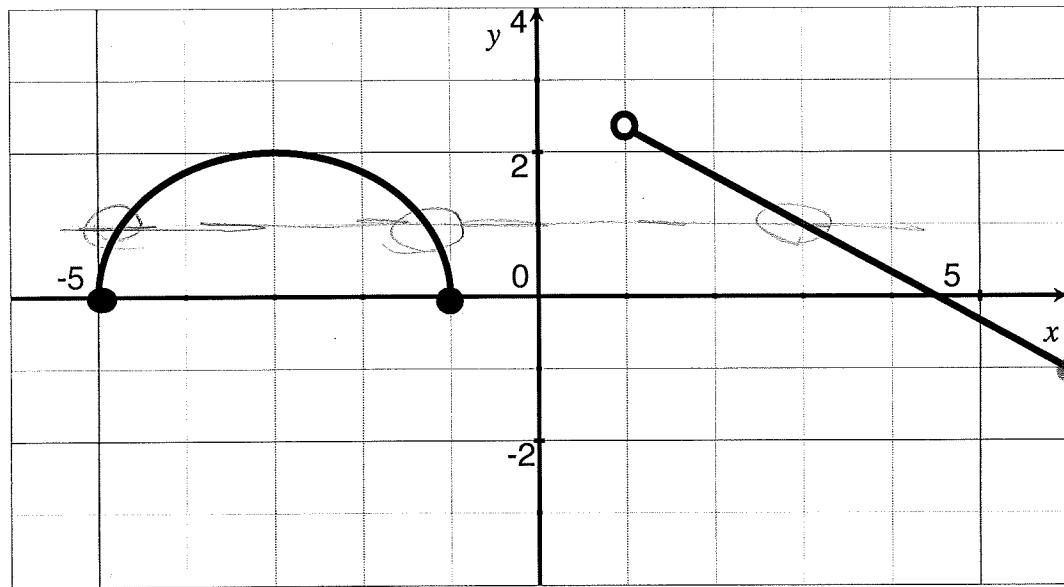
$$\frac{x}{x-5} \neq 0$$

$$\left. \begin{array}{l} \text{consider } \frac{4x}{x-5} = 8 \\ \Rightarrow 4x = 8x - 40 \\ \Rightarrow +4x = +40 \\ \Rightarrow x = 10 \\ \text{smaller } \text{---} \text{ larger than } 8 \end{array} \right\}$$

$$10 \Rightarrow x \geq 10$$

$$\text{so } [10, \infty)$$

4. [3] Let the following be the graph of g .



(a) What is the domain of g ?

$$[-5, -1] \cup (1, 6)$$

(b) The function g is a piecewise defined function consisting of a straight line and a semicircle. Write down the rule for g .

Semicircle w/ center at $(-3, 0)$
and radius 2

$$(x+3)^2 + (y-0)^2 = 2^2$$

$$(x+3)^2 + y^2 = 4$$

$$y = \sqrt{4 - (x+3)^2}$$

(c) Use the graph above to estimate all x value(s) so that $g(x) = 1$

circled above

$$x \approx -4.7, -1.3 \text{ and } 3$$

line w/ slope $-\frac{2}{3}$ So
+ through $(3, 1)$

$$y = -\frac{2}{3}(x+3) + b$$

$$\Rightarrow b = 3$$

$$y = -\frac{2}{3}x + 3$$

$$g(x) = \begin{cases} \sqrt{4 - (x+3)^2} & \text{if } -5 \leq x \leq -1 \\ -\frac{2}{3}x + 3 & \text{if } -1 < x \leq 6 \end{cases}$$

This can be found exactly by
solving for x in
 $1 = \sqrt{4 - (x+3)^2}$ and $1 = -\frac{2}{3}x + 3$

(d) Find the total length (of the curve and the line) that is graphed above.

Total length = length of semicircle + length of line.
semicircle:

$$\frac{1}{2}[2\pi \cdot 2] = 2\pi$$

$$= 2\pi + \sqrt{25 + 100}$$

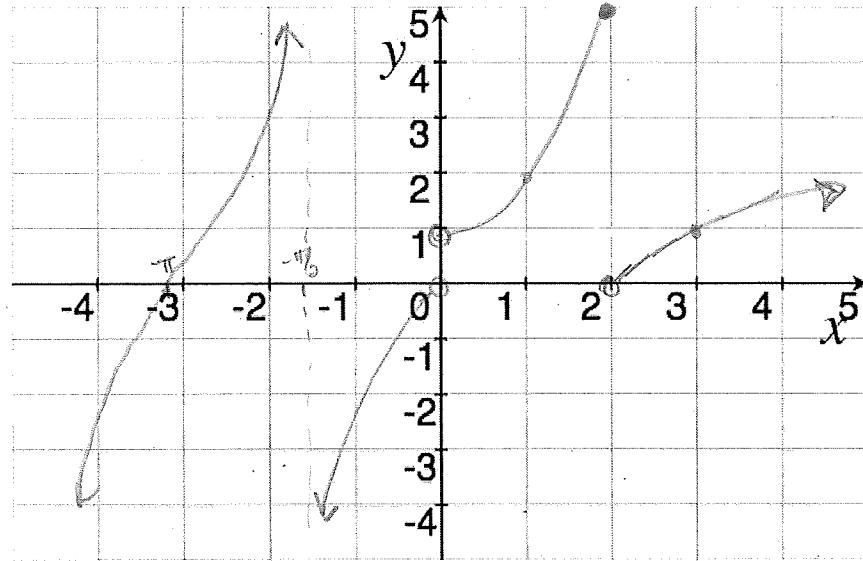
Line: connects $(-1, 1)$ to $(1, \frac{7}{3})$

$$\sqrt{5^2 + (\frac{10}{3})^2}$$

5. Define f by

$$f(x) = \begin{cases} \tan x & \text{if } x < 0 \\ x^2 + 1 & \text{if } 0 < x \leq 2 \\ \log_2(x-1) & \text{if } x > 2 \end{cases}$$

vert shift up 1
shift right 1 unit



(a) [8] Graph f on the axes above.

(b) [9] Find the following if possible:

$$f(1)$$

$$2 \leftarrow 1^2 + 1$$

$$f(2) + f(3) = 5 + 1 = 6$$

$$f(0) = 0^2 + 1 = 1$$

$$f(3) = \log_2(3-1) = \log_2(2) = 1$$

$$f(0)$$

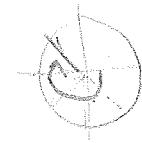
not defined

$$f\left(\frac{-13\pi}{4}\right) = \tan\left(-\frac{13\pi}{4}\right)$$

$$= \tan\left(-\frac{13\pi}{4} + \frac{8\pi}{4}\right) \text{ b/c added a revolution}$$

$$= \tan\left(-\frac{5\pi}{4}\right)$$

$$\therefore \frac{\sin -\frac{5\pi}{4}}{\cos -\frac{5\pi}{4}}$$



Range of f

$$\mathbb{R} \text{ or } (-\infty, \infty)$$

$$\therefore \frac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = -1$$

6. [8] Find all of the exact values x that satisfy the following:

$$2 \sin x = -\sqrt{3}$$

$$\sin x = -\frac{\sqrt{3}}{2}$$



$x = -60^\circ$ or terminal angles with -60°

$\frac{90^\circ}{2}$

$x = -120^\circ$ or terminal angles with -120°

$$5^{4x-1} = 7^x$$

$$\ln 5^{4x-1} = \ln 7^x$$

$$(4x-1) \ln 5 = x \ln 7$$

$$4x \ln 5 - \ln 5 = x \ln 7$$

$$4x \ln 5 - x \ln 7 = \ln 5$$

$$x(4 \ln 5 - \ln 7) = \ln 5$$

$$x = \frac{\ln 5}{4 \ln 5 - \ln 7}$$

7. [4] Find all exact values for x that satisfy the following:

$$\log(x-16) = 2 - \log(x-1)$$

$$\log(x-16) + \log(x-1) = 2$$

$$\log((x-16)(x-1)) = 2$$

$$(x-16)(x-1) = 100$$

$$x^2 - 17x + 16 = 100$$

$$x^2 - 17x - 84 = 0$$

$$(x-21)(x+4) = 0$$

domain problems
doesn't work in original problem

$\Rightarrow x = 21$ ~~or~~

~~assume c > 0~~

$$\frac{\sqrt{c^2 d^6}}{\sqrt{4c^3 d^{-4}}} = \frac{(c^2 d^6)^{\frac{1}{2}}}{(4c^3 d^{-4})^{\frac{1}{2}}}$$

$$= \frac{(c^2)^{\frac{1}{2}} (d^6)^{\frac{1}{2}}}{4^{\frac{1}{2}} (c^3)^{\frac{1}{2}} (d^{-4})^{\frac{1}{2}}} = \frac{|c| |d|^3}{2 c^{\frac{3}{2}} d^{-2}}$$

$$= \frac{1}{2} |c|^{1-\frac{3}{2}} |d|^{3-2}$$

$$= \frac{1}{2} |c|^{\frac{1}{2}} |d|^5$$

$$3^{5x} 9^x = 27$$

$$3^{5x} (3^2)^x = 3^3$$

$$3^{5x} 3^{2x} = 3^3$$

$$3^{5x+2x} = 3^3$$

$$\cancel{3^{5x+2x}} = \cancel{3^3}$$

$$5x + 2x = 3$$

$$7x = 3 \Rightarrow x = \frac{3}{7}$$

checks

$$2 - \log_5(25z)$$

$$2 - [\log_5 25 + \log_5 z]$$

$$2 - \log_5 5^2 - \log_5 z$$

$$2 - 2 - \log_5 z$$

$$- \log_5 z$$

9. [7] Given $f(3) = 0$, use the connection between roots and factors to find the other roots of $f(x) = x^4 - 3x^3 - 25x^2 + 75x$

Since $f(3) = 0$, 3 is a root

$\Rightarrow x-3$ is a factor of $f(x)$

So

$$\begin{array}{r} x-3 \quad | \quad x^4 - 3x^3 - 25x^2 + 75x \\ \underline{- (x^4 - 3x^3)} \\ \quad \quad \quad - 25x^2 + 75x \\ \underline{- (-25x^2 + 75x)} \\ \quad \quad \quad 0 \end{array}$$

$$So \frac{x^4 - 3x^3 - 25x^2 + 75x}{x-3} = x^3 - 25x$$

$$\begin{aligned} \Rightarrow x^4 - 3x^3 - 25x^2 + 75x &= (x-3)(x^3 - 25x) \\ &= (x-3)x(x^2 - 25) \\ &= (x-3)x(x+5)(x-5) \end{aligned}$$

So $x-3, x, x+5 + x-5$ are factors

$\Rightarrow 3, 0, -5 + 5$ are roots

10. Simplify:

$$\sin^{-1}(\sin \frac{3\pi}{4}) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = ?$$

$$\frac{\cos x}{1 - \sin x} + \frac{1 - \sin x}{\cos x}$$

$$\Rightarrow \sin ? = \frac{1}{\sqrt{2}} \text{ where } -\frac{\pi}{2} \leq ? \leq \frac{\pi}{2}$$

$$(\cos x)^2 + (1 - \sin x)^2$$

$$\Rightarrow ? = \frac{\pi}{4}$$

$$(\cos x)(1 - \sin x) \quad \text{P.M.}$$

$$\begin{aligned} &[(\cos x)^2 + 1 - 2\sin x + (\cos x)^2] = 1 + 1 - 2\sin x \\ &\cos x(1 - \sin x) \quad \cos x(1 - \sin x) \end{aligned}$$

11. [4] Let $-\frac{\pi}{2} < \theta < 0$ and $\cos \theta = \frac{1}{5}$. Find $\tan \theta$.

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \text{ we need to find } \sin \theta$$

$$= \frac{2 - 2\sin x}{\cos x(1 - \sin x)} = \frac{2(1 - \sin x)}{\cos x(1 - \sin x)}$$

$$\text{Pyth: } (\sin \theta)^2 (\cos \theta)^2 = 1 \quad \Rightarrow \sin \theta = \frac{\sqrt{24}}{5}$$

$$\text{b/c } \sin \theta \text{ is negative ray} = \frac{2}{\cos x}$$

$$(\sin \theta)^2 + \left(\frac{1}{5}\right)^2 = 1$$

$$(\sin \theta)^2 = 1 - \frac{1}{25} = \frac{24}{25} \Rightarrow \tan \theta = \frac{\pm \sqrt{24}}{\pm 1} = \pm \sqrt{24}$$

12. [6] Let $\frac{\pi}{2} < \phi < \pi$ and $-\frac{\pi}{2} < \theta < 0$. Given that $\sin \phi = \frac{2}{3}$ and that $\cos \theta = \frac{1}{5}$, find $\cos(\theta + \phi)$. (You are free to use results from #10 above.)

(exactly)

to find $\cos \phi$, use Pyth.

$$(\cos \phi)^2 + (\sin \phi)^2 = 1$$

$$(\cos \phi)^2 = 1 - \left(\frac{2}{3}\right)^2 = \frac{5}{9}$$

$$\cos \phi = \pm \frac{\sqrt{5}}{3} \quad \text{b/c } \cos \phi \text{ is negative}$$

$$\begin{aligned} \cos(\theta + \phi) &= \cos \theta \cos \phi - \sin \theta \sin \phi \\ &= \frac{1}{5} \cos \phi - \sin \theta \cdot \frac{2}{3} \end{aligned}$$

$$\text{From #11 } \sin \theta = -\frac{\sqrt{24}}{5} \text{ for the right}$$

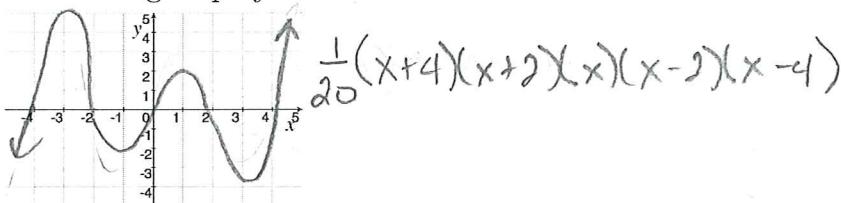
$$\cos \phi = -\frac{\sqrt{5}}{3}$$

$$\cos(\theta + \phi) = \frac{1}{5} \cdot \frac{\sqrt{5}}{3} - \frac{-\sqrt{24}}{5} \cdot \frac{2}{3} = \frac{-\sqrt{5} + 2\sqrt{24}}{15}$$

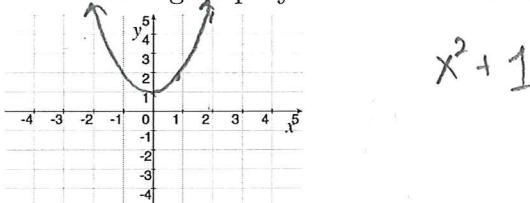
?? There are MANY correct answers for this??

13. Provide a graph AND an algebraic rule for each of the functions described below:

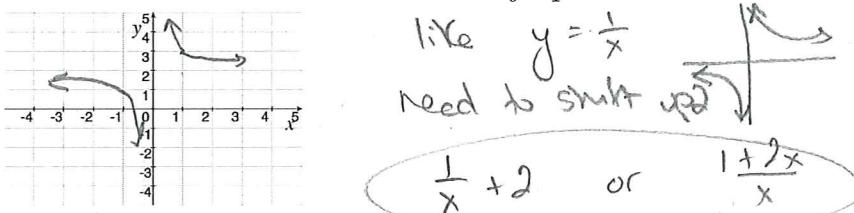
(a) A sixth degree polynomial with 6 distinct roots.



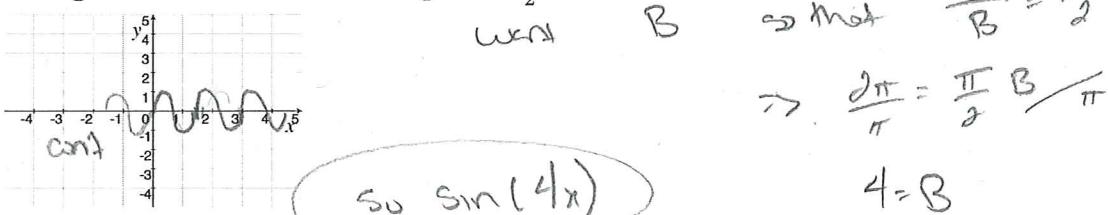
(b) A second degree polynomial with no real roots.



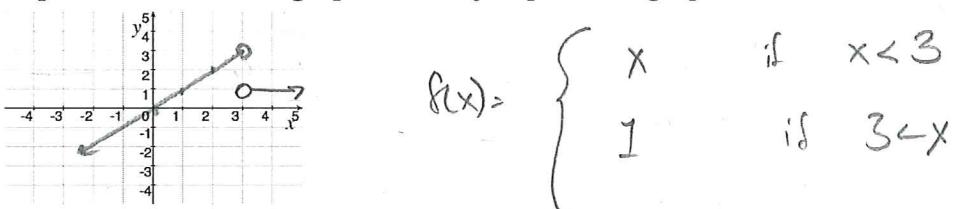
(c) A rational function with a vertical asymptote at $x=2$.



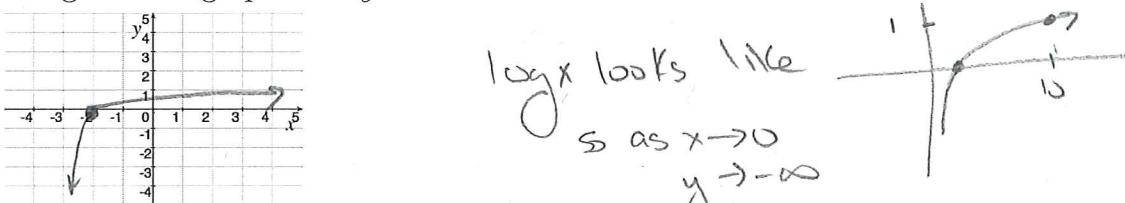
(d) A trigonometric function with period $\frac{\pi}{2}$.



(e) A piece-wise defined graph with a jump in the graph when $x = 3$.



(f) A logarithmic graph with $y \rightarrow -\infty$ when $x \rightarrow -3$.



$$\text{so } \log_{10}(x+3)$$

- 12 [5] You're given a 16 oz mocha that is a rather weak 3% espresso. You, knowing you'll be up late studying mathematics, would rather like a 30% espresso drink. Realizing this you purchase an espresso machine. How much weak mocha do you discard and replace with straight espresso to have a 16 oz mocha with the desired concentration?

let x be the mocha you keep
 y be the amount of espresso

$$\begin{array}{r} 16 \\ - 3 \\ \hline 13 \end{array}$$

$$\text{total mocha} \Rightarrow 16 = x + y$$

$$\text{total espresso} \Rightarrow 3 = .03x + y$$

2 equations, 2 unknowns

$$16 = x + y \quad \text{and} \quad 3 = .03x + y$$

$$\Rightarrow 16 - y = x \quad \text{sub into } 3$$

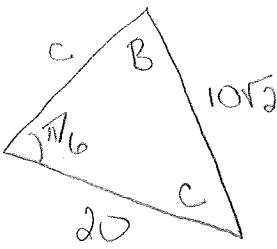
$$\text{to get} \quad 3 = .03(16 - y) + y$$

$$3 = .03(16 - y) + y$$

$$3 = .03(16 - y) + y$$

$$\Rightarrow y = \frac{4.8}{.97} = \frac{4.8}{97} \text{ oz}$$

- 13 [5] Use the conventions from the book and class and let A be measure of the angle opposite the side with length a . Given that $a = 10\sqrt{2}$, $b = 20$, and $A = \frac{\pi}{6}$ with the standard notation, determine if the information describes 0, 1, or 2 triangles and solve for them/it if they/it exist/s.



$$\frac{\sin \frac{\pi}{6}}{10\sqrt{2}} = \frac{\sin B}{20}$$

$$20 \sin \frac{\pi}{6} = \sin B$$

$$\frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} \cdot \frac{1}{2} = \sin B$$

$$\Rightarrow B = \frac{\pi}{4} \quad \text{or} \quad \frac{3\pi}{4}$$

$$\text{if } B = \frac{\pi}{4}$$

$$C = \pi - \frac{\pi}{6} - \frac{\pi}{4} \\ = 12\pi - 3\pi - 2\pi$$

$$\therefore C = \frac{7\pi}{12}$$

$$\text{if } B = \frac{3\pi}{4}$$

$$C = \pi - \frac{3\pi}{4} - \frac{\pi}{6} \\ = 12\pi - 9\pi - 2\pi$$

$$\therefore C = \frac{\pi}{12}$$

$$\Rightarrow \frac{\sin \frac{\pi}{6}}{10\sqrt{2}} = \frac{\sin \frac{\pi}{12}}{c}$$

$$\frac{1}{20\sqrt{2}} = \frac{\sin \frac{\pi}{12}}{c}$$

$$\frac{\sin \frac{\pi}{6}}{10\sqrt{2}} = \frac{\sin \frac{\pi}{12}}{c}$$

$$\frac{1}{20\sqrt{2}} = \frac{\sin \frac{\pi}{12}}{c}$$

$$c = 20\sqrt{2} \sin \frac{\pi}{12}$$

$$c = 20\sqrt{2} \sin \left(\frac{3\pi}{12} + \frac{4\pi}{12} \right)$$

$$c = 20\sqrt{2} \left[\sin \frac{\pi}{4} \cos \frac{\pi}{3} + \cos \frac{\pi}{4} \sin \frac{\pi}{3} \right]$$

$$c = 20\sqrt{2} \sin \frac{\pi}{12}$$

$$c = 20\sqrt{2} \sin \left(\frac{4\pi}{12} - \frac{3\pi}{12} \right)$$

$$c = 20\sqrt{2} \left[\sin \frac{\pi}{3} \cos \frac{\pi}{4} + \cos \frac{\pi}{3} \sin \frac{\pi}{4} \right]$$

$$c = 20\sqrt{2} \left[\frac{1}{2} \frac{1}{2} + \frac{1}{2} \frac{\sqrt{3}}{2} \right] = 10(\sqrt{3}) = 10(\sqrt{3} - 1)$$

Recall

$$\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi$$

15. Suppose a radioactive isotope is such that one-fifth of the atoms in a sample decay after three years. Find the half-life of this isotope

$$\text{use } P_0 \left(\frac{1}{5}\right)^n = P(t) \quad (\text{or one of the other versions?})$$

start with P_0 and end up with $\frac{1}{5}P_0$ when $t=3$

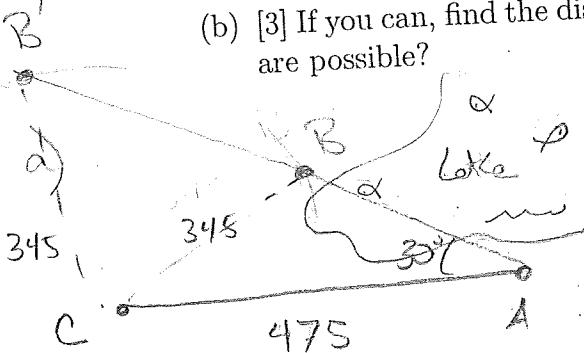
$$\frac{1}{5}P_0 = P_0 \left(\frac{1}{5}\right)^{3/h} \text{ solve for } h$$

$$\begin{aligned} \frac{1}{5} &= \left(\frac{1}{5}\right)^{3/h} \Rightarrow \ln \frac{1}{5} = \ln \left(\frac{1}{5}\right)^{3/h} \Rightarrow \ln \frac{1}{5} = \frac{3}{h} \ln \left(\frac{1}{5}\right) \text{ so} \\ \frac{1}{5} &= \left(\frac{1}{5}\right)^{3/h} \Rightarrow \ln \frac{1}{5} = \ln \left(\frac{1}{5}\right)^{3/h} \Rightarrow \ln \frac{1}{5} = \frac{3}{h} \ln \left(\frac{1}{5}\right) \Rightarrow h = \frac{\ln \frac{1}{5}}{\ln \frac{1}{5}} \end{aligned}$$

16. Points A and B are separated by a lake. To find the distance between them, a surveyor locates a point on land such that $\angle CAB = 30^\circ$. She also measures CA as 475ft and CB as 345ft.

- (a) [2] Draw a picture of the situation. Do you have enough information to find the distance between A and B? Justify yourself.

- (b) [3] If you can, find the distance between A and B. If you can't, what two distances are possible?



We don't?

This is a SSA or ASS situation and there could

be 0, 1, or 2 triangles that

satisfy this information

$$\left. \begin{array}{l} \text{if } \angle B = .759 \text{ rad or } 43.5^\circ \\ \Rightarrow \angle C = 106^\circ \end{array} \right\} \quad \left. \begin{array}{l} \text{if } \angle B = 2.303 \text{ rad or } 136.5^\circ \\ \Rightarrow \angle C = 13.5^\circ \end{array} \right\}$$

$$\left. \begin{array}{l} \sin 106^\circ = \frac{\sin 30^\circ}{345} \\ ? \end{array} \right\}$$

$$\Rightarrow ? = \left(\frac{\sin 106^\circ}{\sin 30^\circ} \right) 345 \approx 663.3 \text{ ft}$$

b) law of sines

$$\frac{\sin 30^\circ}{345} = \frac{\sin B}{475}$$

$$\Rightarrow .6883 \frac{.475}{345} = \sin B$$

$$\Rightarrow B = .759 \text{ rad or } 43.5^\circ$$

$$\begin{aligned} \text{or } B &= \pi - .759 \text{ rad or } 180 - 43.5^\circ \\ &= 2.363 \text{ rad or } 136.5^\circ \end{aligned}$$

So we look at two cases:

$$\left. \begin{array}{l} \text{so } \frac{\sin 13.5^\circ}{?} = \frac{\sin 30^\circ}{345} \\ ? \end{array} \right\}$$

$$\Rightarrow ? = \left(\frac{\sin 13.5^\circ}{\sin 30^\circ} \right) 345 \approx 161.1 \text{ ft}$$