

## Practice

## TMath 120

## Final

NAME: This is a sample final to be used for practice. This is *not a template* for the Final that will be given in class. Many of the questions on the Final will look quite different than those appearing here.

[10] Let  $f$  &  $g$ , be functions.

T  (F)  $(f \circ g)(x) = (g \circ f)(x)$  ex  $f(x) = x+1$   $g(x) = 3x$

T  (F)  $(\frac{f}{g})(x) = (\frac{g}{f})(x)$   $(f \circ g)(x) = 3x+1 \neq 3(x+1) = (g \circ f)(x)$

T  (F)  $\sqrt{(x^2)} = x$  for all real numbers  $x$ . Let  $x = -2$

(T) F If 2 is a root of  $g$ , then  $g(2) = 0$ . by definition of root

(T) F  $\ln \frac{x}{y} = \ln x - \ln y$  for all positive numbers  $x$  and  $y$ .

(T) F  $\log(\log(10)) = 0$ .  $\log 10 = 1 \Rightarrow \log(\log(10)) - \log(1) = 0$  ✓

T  (F)  $\sin^{-1}(\sin x) = x$  for all real numbers  $x$ . Let  $x = \frac{3\pi}{4}$

T  (F)  $\sin(\frac{\pi}{3} + x) = \sin \frac{\pi}{3} + x$  PEMDAS  $\Rightarrow \sin \frac{\pi}{3} + x = (\sin \frac{\pi}{3}) + x$

(T) F If  $\sin \theta \neq 0$  and  $\tan \theta < 0$ , then  $\cos \theta < 0$   $\begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array}$

T  (F) The range of  $\sin^{-1}$  is  $[0, \pi]$

Right answers will *not* get credit without supporting work. Note "undefined" and "no solution" are possible answers.

1. Find all  $x$  such that

$$2(5 - (8 - x)^2)^{-\frac{1}{2}} - 1 = 0$$

$$\begin{aligned} \frac{2}{\sqrt{5 - (8 - x)^2}} - 1 &= 0 \\ \frac{2}{\sqrt{5 - (8 - x)^2}} &= 1 \\ 2 &= \sqrt{5 - (8 - x)^2} \\ 4 &= 5 - (8 - x)^2 \end{aligned}$$

$\Rightarrow -1 = -(8 - x)^2$   
 $1 = (8 - x)^2$   
 $\pm\sqrt{1} = 8 - x$   
 $\Downarrow$   
 $1 = 8 - x \quad \text{or} \quad -1 = 8 - x$   
 $x = 7 \quad \text{or} \quad x = 9$

2. [2] Explain what a function is.

A function consists of 2 sets (a domain & a range) and a rule between them such that every number in the domain is sent to exactly one output.

3. Given  $m(x) = \frac{x}{x-5}$ , and  $n(x) = \sqrt{4x-8}$ ,

(a) The function  $m$  passes the horizontal line test. Find  $m^{-1}$ .

$$m(m^{-1}(x)) = x \quad \text{or} \quad x = \frac{y}{y-5} \quad \begin{aligned} xy - y &= 5x \\ y(x-1) &> 5x \\ y &= \frac{5x}{x-1} \end{aligned}$$

$$\Rightarrow x = \frac{m^{-1}(x)}{m^{-1}(x)-5} \quad \begin{aligned} \Rightarrow x(y-5) &= y \\ \Rightarrow xy - 5x &= y \end{aligned}$$

(b) [4] If  $p(x) = 3m(x+1)$ , find the domain and rule of  $p$ .

$$p(x) = 3m(x+1)$$

$$= 3 \frac{|x+1|}{|x+1|-5}$$

$$= 3 \left( \frac{x+1}{x-4} \right)$$

domain: all  $x$  so that denominator  $\neq 0$

$$x-4 \neq 0$$

$$x \neq 4$$

so  $(-\infty, 4)$  and  $(4, \infty)$

(c) [3] Find the domain and rule of  $n \circ m$ .

$$n \circ m(x) = n\left(\frac{x}{x-5}\right)$$

$$= \sqrt{4\left|\frac{x}{x-5}\right| - 8}$$

domain: all  $x$  so that denominator  $\neq 0$  and  $sgt \geq 0$

$$x-5 \neq 0$$

$$\frac{4x}{x-5} - 8 \geq 0$$

$$x \neq 5$$

$$\frac{4x}{x-5} \geq 8$$

(d) [5] Find the domain and rule of  $\frac{n}{m}$ .

$$\left(\frac{n}{m}\right)(x) = \frac{n(x)}{m(x)} \quad \text{so } (-\infty, 0) \cup (0, 5) \cup (5, \infty)$$

$$= \frac{\sqrt{4x-8}}{\left(\frac{x}{x-5}\right)}$$

consider  $\frac{4x}{x-5} = 8$

$$\Rightarrow 4x = 8x - 40$$

$$\Rightarrow +4x = +40$$

$$\Rightarrow x = 10$$

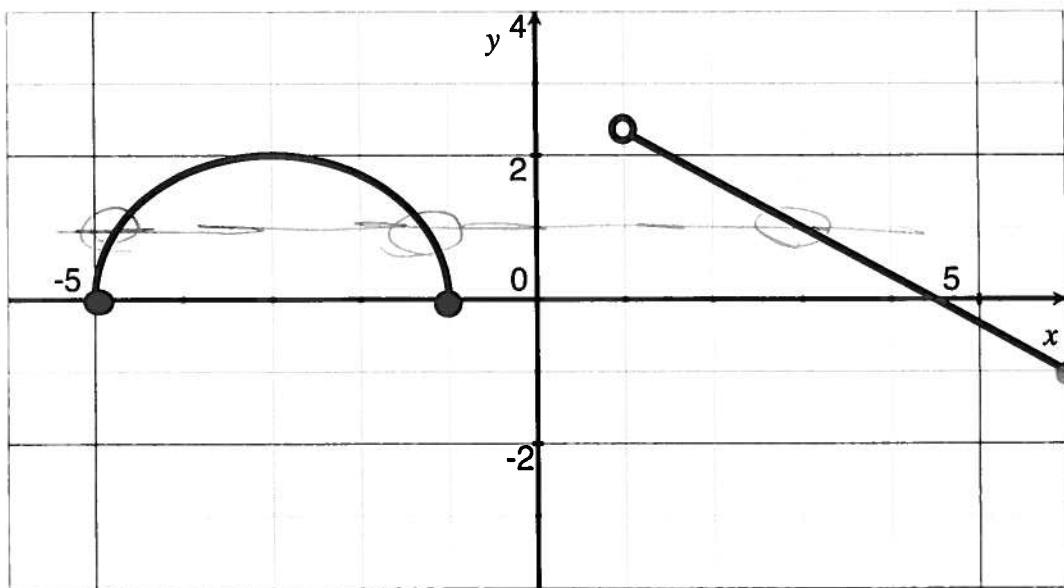
smaller num  $\textcircled{a}$  larger num  $\textcircled{b}$

$$10 \Rightarrow x \geq 10$$

so  $[10, \infty)$

domain: all #'s so that  
den of den  $\neq 0$  and dent  $\neq 0$   
 $x-5 \neq 0$        $\frac{x}{x-5} \neq 0$

4. [3] Let the following be the graph of  $g$ .



- (a) What is the domain of  $g$ ?

$$[-5, -1] \cup (1, 6)$$

- (b) The function  $g$  is a piecewise defined function consisting of a straight line and a semicircle. Write down the rule for  $g$ .

Semicircle w/ center at  $(-3, 0)$   
and radius 2

$$(x + 3)^2 + (y - 0)^2 = 2^2$$

$$(x + 3)^2 + y^2 = 4$$

$$y = \sqrt{4 - (x + 3)^2}$$

line w/ slope  $-\frac{2}{3}$   
+ through  $(3, 1)$

$$l = -\frac{2}{3}(3) + b$$

$$\Rightarrow b = 3$$

$$y = -\frac{2}{3}x + 3$$

$$g(x) = \begin{cases} \sqrt{4 - (x+3)^2} & \text{if } -5 \leq x \leq -1 \\ -\frac{2}{3}x + 3 & \text{if } -1 < x \leq 6 \end{cases}$$

- (c) Use the graph above to estimate all  $x$  value(s) so that  $g(x) = 1$ ?

circled above

$$x \approx -4.7, -1.3 \text{ and } 3$$

This can be found exactly by  
solving for  $x$  in  
 $1 = \sqrt{4 - (x+3)^2}$  and  $1 = -\frac{2}{3}x + 3$

- (d) Find the total length (of the curve and the line) that is graphed above.

Total length = length of semicircle + length of line  
semicircle:  
 $\frac{1}{2}[2\pi \cdot 2] = 2\pi$

$$= 2\pi + \sqrt{25 + 16}$$

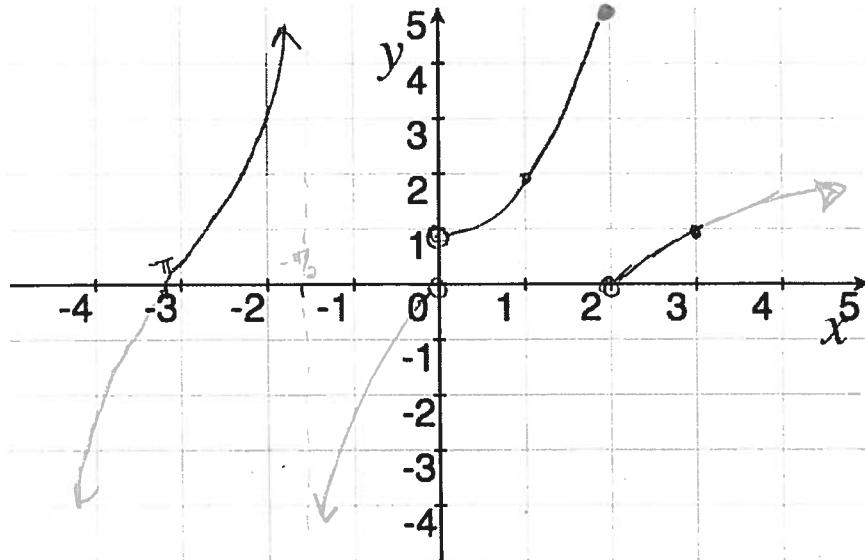
line: connects  $(6, 1)$  to  $(1, \frac{7}{3})$

$$\sqrt{5^2 + (4\frac{1}{3})^2}$$

5. Define  $f$  by

$$f(x) = \begin{cases} \tan x & \text{if } x < 0 \\ x^2 + 1 & \text{if } 0 < x \leq 2 \\ \log_2(x-1) & \text{if } 2 < x \end{cases}$$

vert shift up 1  
shift right 1 unit.



(a) [8] Graph  $f$  on the axes above.

(b) [9] Find the following if possible:

$$f(1)$$

$$2 \quad \lceil 1^2 + 1 \rfloor$$

$$f(2) + f(3) = 5 + 1 = 6$$

$$f(2) = 2^2 + 1 = 5$$

$$f(3) = \log_2(3-1) = \log_2(2) = 1$$

$$f(0)$$

not defined

$$f\left(-\frac{13\pi}{4}\right) = \tan\left(-\frac{13\pi}{4}\right)$$

$$= \tan\left(-\frac{13\pi}{4} + \frac{8\pi}{4}\right) \text{ b/c added a revolution}$$

$$= \tan\left(-\frac{5\pi}{4}\right)$$

$$= \frac{\sin -\frac{5\pi}{4}}{\cos -\frac{5\pi}{4}}$$



Range of  $f$

$$\mathbb{R} \propto (-\infty, \infty)$$

$$= \frac{1}{\sqrt{2}} = -1$$

6. [8] Find all of the exact values  $x$  that satisfy the following:

$$2 \sin x = -\sqrt{3}$$

$$\sin x = -\frac{\sqrt{3}}{2}$$



$x = -60^\circ$  or coterminal angles  
with  $-60^\circ$

$90^\circ$

$x = -120^\circ$  or coterminal angles  
with  $-120^\circ$

$$5^{4x-1} = 7^x$$

$$\ln 5^{4x-1} = \ln 7^x$$

$$(4x-1)\ln 5 = x \ln 7$$

$$x^4 \ln 5 - \ln 5 = x \ln 7$$

$$x^4 \ln 5 - x \ln 7 = \ln 5$$

$$x(4 \ln 5 - \ln 7) = \ln 5$$

$$x = \frac{\ln 5}{4 \ln 5 - \ln 7}$$

7. [4] Find all exact values for  $x$  that satisfy the following:

$$\log(x-16) = 2 - \log(x-1)$$

$$\log(x-16) + \log(x-1) = 2$$
 ~~$\log(x-16)(x-1) = 2$~~

$$(x-16)(x-1) = 100$$

$$x^2 - 17x + 16 = 100$$

$$x^2 - 17x - 84 = 0$$

$$(x-21)(x+4) = 0$$

↓ domain problem  
doesn't work in original problem

$$\Rightarrow x = 21 \quad \text{or} \quad \cancel{x = -4}$$

8. Simplify:

$$\frac{\sqrt{c^2 d^6}}{\sqrt{4c^3 d^{-4}}} = \frac{(c^2 d^6)^{\frac{1}{2}}}{(4c^3 d^{-4})^{\frac{1}{2}}}$$

$$= \frac{(c^2)^{\frac{1}{2}} (d^6)^{\frac{1}{2}}}{4^{\frac{1}{2}} (c^3)^{\frac{1}{2}} (d^{-4})^{\frac{1}{2}}} = \frac{|c| |d|^3}{2 c^{\frac{3}{2}} d^{-2}}$$

$$= \frac{1}{2} |c|^{1-\frac{3}{2}} |d|^{3-2}$$

$$= \frac{1}{2} |c|^{\frac{1}{2}} |d|^5$$

$$3^{5x} 9^x = 27$$

$$3^{5x} (3^2)^x = 3^3$$

$$3^{5x} 3^{2x} = 3^3$$

$$3^{5x+2x} = 3^3$$

$$\log 3^{5x+2x} = \log 3^3$$

$$5x+2x = 3$$

$$7x = 3 \Rightarrow x = \frac{3}{7}$$

✓ checks

$$2 - \log_5(25z)$$

$$2 - [\log_5 25 + \log_5 z]$$

$$2 - \log_5 5^2 - \log_5 z$$

$$2 - 2 - \log_5 z$$

$$- \log_5 z$$

9. [7] Given  $f(3) = 0$ , use the connection between roots and factors to find the other roots of  $f(x) = x^4 - 3x^3 - 25x^2 + 75x$

Since  $f(3) = 0$ ,  $3$  is a root

$\Rightarrow x-3$  is a factor of  $f(x)$

So

$$\begin{array}{r} x^3 - 25x \quad \text{RQ} \\ x-3 \boxed{x^4 - 3x^3 - 25x^2 + 75x} \\ - (x^4 - 3x^3) \\ \hline - 25x^2 + 75x \\ - (-25x^2 + 75x) \\ \hline 0 \end{array}$$

$$\text{So } \frac{x^4 - 3x^3 - 25x^2 + 75x}{x-3} = x^3 - 25x$$

$$\begin{aligned} &\Rightarrow x^4 - 3x^3 - 25x^2 + 75x = (x-3)(x^3 - 25x) \\ &= (x-3)x(x^2 - 25) \\ &= (x-3)x(x+5)(x-5) \end{aligned}$$

So  $x-3, x, x+5 + x-5$  are factors

$\Rightarrow 3, 0, -5 + 5$  are roots

10. Simplify:

$$\sin^{-1}(\sin \frac{3\pi}{4}) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = ?$$

$$\Rightarrow \sin ? = \frac{1}{\sqrt{2}} \text{ where } -\frac{\pi}{2} \leq ? \leq \frac{\pi}{2}$$

$$\Rightarrow ? = \frac{\pi}{4}$$

11. [4] Let  $\frac{\pi}{2} < \theta < 0$  and  $\cos \theta = \frac{1}{5}$ . Find  $\tan \theta$ .

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \text{ we need to find } \sin \theta$$

$$\text{Given: } (\sin \theta)^2 + (\cos \theta)^2 = 1 \quad \Rightarrow \sin \theta = \frac{\sqrt{24}}{5}$$

$$(\sin \theta)^2 + \left(\frac{1}{5}\right)^2 = 1$$

$$(\sin \theta)^2 = 1 - \frac{1}{25} = \frac{24}{25} \Rightarrow \sin \theta = \frac{-\sqrt{24}}{5} = -\frac{\sqrt{24}}{5}$$

12. [6] Let  $\frac{\pi}{2} < \phi < \pi$  and  $\frac{-\pi}{2} < \theta < 0$ . Given that  $\sin \phi = \frac{2}{3}$  and that  $\cos \theta = \frac{1}{5}$ , find  $\cos(\theta + \phi)$ . (You are free to use results from #10 above.)

exactly

$$\begin{aligned} \cos(\theta + \phi) &= \cos \theta \cos \phi - \sin \theta \sin \phi \\ &= \frac{1}{5} \cos \phi - \sin \theta \cdot \frac{2}{3} \end{aligned}$$

$$\text{From } \#11 \quad \sin \theta = -\frac{\sqrt{24}}{5} \text{ from the right}$$

$$\cos \phi = -\frac{\sqrt{5}}{3}$$

$$\cos(\theta + \phi) = \frac{1}{5} \cdot \frac{\sqrt{5}}{3} - \frac{\sqrt{24}}{5} \cdot \frac{2}{3} = \frac{-\sqrt{15} + 2\sqrt{24}}{15}$$

$$\frac{\cos x}{1 - \sin x} + \frac{1 - \sin x}{\cos x}$$

$$(\cos x)^2 + (1 - \sin x)^2$$

$$(\cos x)(1 - \sin x) \quad \text{Pythag.}$$

$$\frac{(\cos x)^2 + 1 - 2\sin x + (\cos x)^2}{\cos x(1 - \sin x)} = \frac{1 + 1 - 2\sin x}{\cos x(1 - \sin x)}$$

$$= \frac{2 - 2\sin x}{\cos x(1 - \sin x)} = \frac{2(1 - \sin x)}{\cos x(1 - \sin x)}$$

$$= \frac{2}{\cos x}$$

To find  $\cos \phi$ , use Pythag.

$$(\cos \phi)^2 + (\sin \phi)^2 = 1$$

$$(\cos \phi)^2 = 1 - \left(\frac{2}{3}\right)^2 = \frac{5}{9}$$

$$\cos \phi = \frac{\sqrt{5}}{3}$$

 cos neg

- 13 [5] You're given a 16 oz mocha that is a rather weak 3% espresso. You, knowing you'll be up late studying mathematics, would rather like a 30% espresso drink. Realizing this you purchase an espresso machine. How much weak mocha do you discard and replace with straight espresso to have a 16 oz mocha with the desired concentration?

let  $x$  be the mocha you keep  
 $y$  be the amount of espresso

$$\begin{array}{c} 16 \\ \hline x+y \end{array}$$

$$\text{total mocha} \Rightarrow 16 = x + y$$

$$\text{total espresso} \Rightarrow 3.16 = .03x + y$$

2 equations, 2 unknowns

$$16 = x + y \quad \text{and} \quad 3.16 = .03x + y$$

$$\Rightarrow 16 - y = x \quad \text{sub into } \rightarrow$$

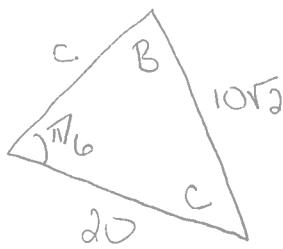
$$\text{to get} \quad 4.16 = .03(16-y) + y$$

$$\begin{aligned} 4.16 &= .03(16-y) + y \\ -4.16 &- .03(16-y) \end{aligned}$$

$$4.16 = .97y$$

$$\Rightarrow y = \frac{4.16}{.97} = \frac{4.16}{.97} \text{ oz}$$

13. [5] Use the conventions from the book and class and let  $A$  be measure of the angle opposite the side with length  $a$ . Given that  $a = 10\sqrt{2}$ ,  $b = 20$ , and  $A = \frac{\pi}{6}$  with the standard notation, determine if the information describes 0, 1, or 2 triangles and solve for them/it if they/it exist/s.



$$\begin{aligned} \frac{\sin \frac{\pi}{6}}{10\sqrt{2}} &= \frac{\sin B}{20} \\ 20 \sin \frac{\pi}{6} &= \sin B \\ \frac{20 \sin \frac{\pi}{6}}{10\sqrt{2}} &= \sin B \end{aligned}$$

$$\boxed{B = \pi/2}$$

$$\begin{aligned} C &= \pi - \pi/2 - \pi/6 \\ &= \frac{12\pi - 3\pi - 2\pi}{12} \\ &\therefore C = 7\pi/12 \end{aligned}$$

$$\begin{aligned} \text{if } B &= 3\pi/4 \\ C &= \pi - 3\pi/4 - \pi/6 \\ &= \frac{12\pi - 9\pi - 2\pi}{12} \\ &\therefore C = \pi/12 \end{aligned}$$

$$\begin{aligned} \frac{1}{\sqrt{2}} &= \frac{2}{\sqrt{2}} \cdot \frac{1}{2} = \sin B \\ \Rightarrow B &= \pi/4 \quad \text{or} \quad 3\pi/4 \end{aligned}$$

$$\begin{aligned} \frac{\sin \frac{\pi}{6}}{10\sqrt{2}} &= \frac{\sin \frac{7\pi}{12}}{C} \\ \frac{1}{20\sqrt{2}} &= \frac{\sin \frac{7\pi}{12}}{C} \end{aligned}$$

$$\begin{aligned} \frac{\sin \frac{\pi}{6}}{10\sqrt{2}} &= \frac{\sin \frac{\pi}{6}}{C} \\ \frac{1}{20\sqrt{2}} &= \frac{\sin \frac{\pi}{6}}{C} \end{aligned}$$

Recall  
 $\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi$

$$\begin{aligned} C &= 20\sqrt{2} \sin \frac{7\pi}{12} & C &= 20\sqrt{2} \sin \frac{\pi}{12} \\ C &= 20\sqrt{2} \sin \left( \frac{3\pi}{12} + \frac{4\pi}{12} \right) & C &= 20\sqrt{2} \sin \left( \frac{4\pi}{12} - \frac{3\pi}{12} \right) \\ C &= 20\sqrt{2} \left[ \sin \frac{\pi}{4} \cos \frac{\pi}{3} + \cos \frac{\pi}{4} \sin \frac{\pi}{3} \right] & C &= 20\sqrt{2} \left[ \sin \frac{\pi}{3} \cos \frac{\pi}{4} + \cos \frac{\pi}{3} \sin \frac{\pi}{4} \right] \\ \boxed{C = 20\sqrt{2} \left[ \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \right]} &= 10(1+\sqrt{3}) & \boxed{C = 10(\sqrt{3}-1)} \end{aligned}$$

15. Suppose a radioactive isotope is such that one-fifth of the atoms in a sample decay after three years. Find the half-life of this isotope

use  $R_0 \left(\frac{1}{5}\right)^{\frac{t}{h}} = R(t)$  (or one of the other versions?)

start with  $R_0$  and end up with  $\frac{1}{5}R_0$  when  $t=3$

$$\frac{1}{5}R_0 = R(3) = R_0 \left(\frac{1}{5}\right)^{\frac{3}{h}}$$

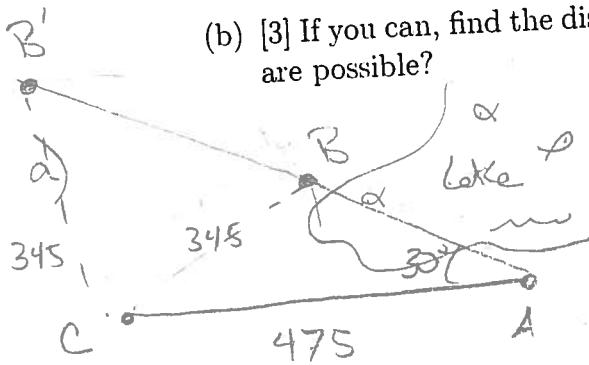
solve for  $h$ ...

$$\begin{aligned} \frac{1}{5} &= \left(\frac{1}{5}\right)^{\frac{3}{h}} \quad \ln \frac{1}{5} = \frac{3}{h} \ln \frac{1}{5} \quad h = \frac{\ln \frac{1}{5}}{\ln \frac{1}{5}} \\ \frac{1}{5} &= \left(\frac{1}{2}\right)^{\frac{3}{h}} \quad \ln \frac{1}{5} = \ln \left(\frac{1}{2}\right)^{\frac{3}{h}} \quad \ln \ln \frac{1}{5} = \ln \frac{1}{2} \end{aligned}$$

16. Points A and B are separated by a lake. To find the distance between them, a surveyor locates a point on land such that  $\angle CAB = 30^\circ$ . She also measures CA as 475ft and CB as 345ft.

(a) [2] Draw a picture of the situation. Do you have enough information to find the distance between A and B? Justify yourself.

(b) [3] If you can, find the distance between A and B. If you can't, what two distances are possible?



We don't?

this is a SSA or ASS situation and there could

be 0, 1, or 2 triangles that

satisfy this information?

b) law of sines

$$\frac{\sin 30^\circ}{345} = \frac{\sin B}{475}$$

$$\Rightarrow 0.688 \approx \frac{\frac{1}{2} \cdot 475}{345} = \sin B$$

$$\Rightarrow B = .759 \text{ rad} \text{ or } 43.5^\circ$$

$$\begin{aligned} \text{or } B &= \pi - .759 \text{ rad} \text{ or } 180 - 43.5^\circ \\ &= 2.363 \text{ rad} \text{ or } 136.5^\circ \end{aligned}$$

So we look at two cases:

if  $\angle B = .759 \text{ rad}$  or  $43.5^\circ$

$$\Rightarrow \angle C = 106^\circ$$

$$\text{so } \frac{\sin 106^\circ}{?} = \frac{\sin 30^\circ}{345}$$

$$\Rightarrow ? = \left( \frac{\sin 106^\circ}{\sin 30^\circ} \right) 345 \approx 663.3 \text{ ft}$$

if  $\angle B = 2.363 \text{ rad}$  or  $136.5^\circ$

$$\Rightarrow \angle C = 13.5^\circ$$

$$\text{so } \frac{\sin 13.5^\circ}{?} = \frac{\sin 30^\circ}{345}$$

$$\Rightarrow ? = \left( \frac{\sin 13.5^\circ}{\sin 30^\circ} \right) 345 \approx 161.1 \text{ ft}$$