Still typo's pg 5 + pg 6

Practice TMath 120 Final

NAME: This is a sample final to be used for practice. This is *not a template* for the Final that will be given in class. Many of the questions on the Final will look quite di?erent than those appearing here.

[10] Let f & g, be functions.

T F $(f \circ g)(x) = (g \circ f)(x)$ ex $f(x) = x+1$ $g(x) = 3x$

T F $(\frac{f}{g})(x) = (\frac{g}{f})(x)$ $(f \circ g)(x) = 3x+1 \neq 3(x+1) = (g \circ f)(x)$

T F $\sqrt{(x^2)} = x$ for all real numbers x. Let x be -2

T F If 2 is a root of g, then $g(2) = 0$. by definition of root

T F $\ln \frac{x}{y} = \ln x - \ln y$ for all positive numbers x and y.

T F $\log(\log(10)) = 0$. $\log 10 = 1 \Rightarrow \log(\log(10)) = \log(1) = 0$ ✓

T F $\sin^{-1}(\sin x) = x$ for all real numbers x. let $x = 3\pi/4$

T F $\sin(\frac{\pi}{3} + x) = \sin\frac{\pi}{3} + x$ PEMDAS $\Rightarrow \sin \pi/3 + x = (\sin \pi/3) + x$

T F If $\sin\theta > 0$ and $\tan\theta < 0$, then $\cos\theta < 0$

sin quad II

T F The range of $\sin^{-1}$ is $[0, \pi]$

range is $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Right answers will *not* get credit without supporting work. Note "undefined" and "no solution" are possible answers.

1. Find all x such that

$$2(5-(8-x)^2)^{-\frac{1}{2}} - 1 = 0$$

$$\frac{2}{\sqrt{5-(8-x)^2}} - 1 = 0$$

$$\frac{2}{\sqrt{5-(8-x)^2}} = 1$$

$$2 = \sqrt{5-(8-x)^2}$$

$$4 = 5-(8-x)^2$$

$$-1 = -(8-x)^2$$

$$1 = (8-x)^2$$

$$\pm\sqrt{1} = 8-x$$

$$1 = 8-x \text{ or } -1 = 8-x$$

$$\Rightarrow x = 7 \text{ or } x = 9$$

2. [2] Explain what a function is.

A function consists of 2 sets (a domain & a range) and a rule between them such that every number in the domain is sent to exactly one output.

3. Given $m(x) = \dfrac{x}{x-5}$, and $n(x) = \sqrt{4x-8}$,

(a) The function m passes the horizontal line test. Find m^{-1}.

$m(m^{-1}(x)) = x$

$\Rightarrow \quad x = \dfrac{m^{-1}(x)}{m^{-1}(x)-5}$

or $x = \dfrac{y}{y-5}$

$\Rightarrow x(y-5) = y$

$\Rightarrow xy - 5x = y$

$xy - y = 5x$

$y(x-1) = 5x$

$y = \dfrac{5x}{x-1}$

(b) [4] If $p(x) = 3m(x+1)$, find the domain and rule of p.

$p(x) = 3m(\boxed{x+1})$

$= 3\dfrac{\boxed{x+1}}{\boxed{x+1}-5}$

$= 3\left(\dfrac{x+1}{x-4}\right)$

domain: all x so that denominator $\neq 0$

$x-4 \neq 0$

$x \neq 4$

so $(-\infty, 4)$ and $(4, \infty)$

(c) [3] Find the domain and rule of $n \circ m$.

$n \circ m(x) = n\left(\boxed{\dfrac{x}{x-5}}\right)$

$= \sqrt{4\boxed{\dfrac{x}{x-5}} - 8}$

domain: all x so that denominator $\neq 0$ and sqrt ≥ 0

$x - 5 \neq 0$

$x \neq 5$

$\dfrac{4x}{x-5} - 8 \geq 0$

$\dfrac{4x}{x-5} \geq 8$

consider $\dfrac{4x}{x-5} = 8$

$\Rightarrow 4x = 8x - 40$

$\Rightarrow -4x = -40$

$\Rightarrow x = 10$

smaller than 8 ~~~~ larger than 8

10 $\Rightarrow x \geq 10$

so $[10, \infty)$

(d) [5] Find the domain and rule of $\dfrac{n}{m}$.

$\left(\dfrac{n}{m}\right)(x) = \dfrac{n(x)}{m(x)}$

$= \dfrac{\sqrt{4x-8}}{\left(\frac{x}{x-5}\right)}$

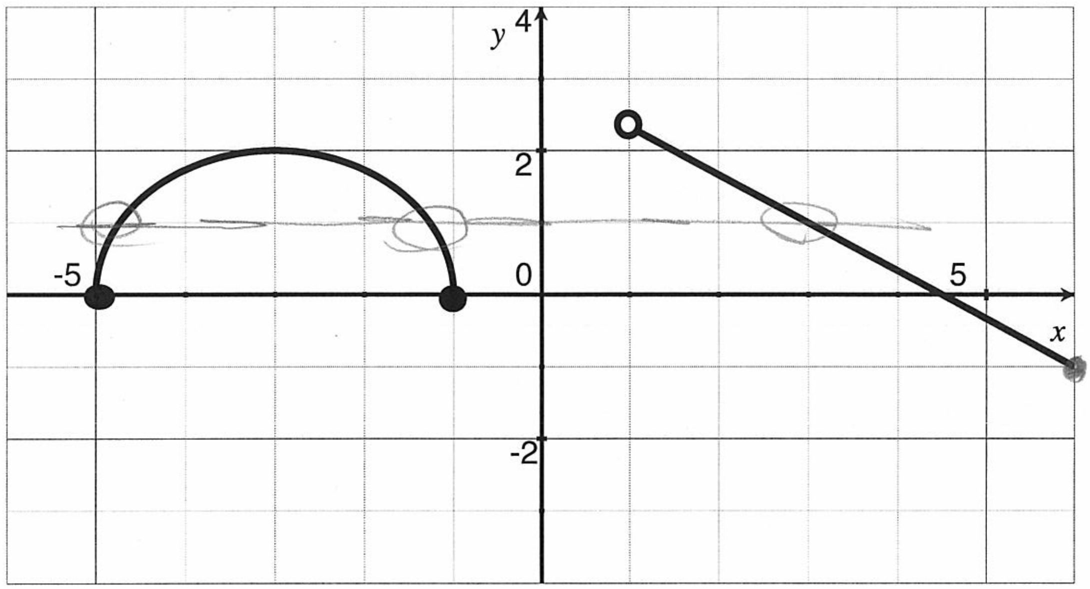
So $(-\infty, 0) \cup (0, 5) \cup (5, \infty)$

domain: all #s so that den of den $\neq 0$ and den $\neq 0$

$x - 5 \neq 0$

$\dfrac{x}{x-5} \neq 0$

4. [3] Let the following be the graph of g.



(a) What is the domain of g?

$[-5,-1] \cup (1,6)$

(b) The function g is a piecewise defined function consisting of a straight line and a semicircle. Write down the rule for g.

Semicircle w/ center at $(-3,0)$ and radius 2

$(x--3)^2+(y-0)^2=2^2$

$(x+3)^2+y^2=4$

$y=\sqrt{4-(x+3)^2}$

line w/ slope $-2/3$ & through $(3,1)$

$1=-\frac{2}{3}(3)+b$

$\leadsto b=3$

$y=-\frac{2}{3}x+3$

So

$g(x)=\begin{cases}\sqrt{4-(x+3)^2} & \text{if } -5\le x\le \\ -\frac{2}{3}x+3 & \text{if } 1<x\le\end{cases}$

(c) Use the graph above to *estimate* all x value(s) so that $g(x)=1$?

Circled above

$x \approx -4.7, -1.3$ and 3

This can be found exactly by solving for x in $1=\sqrt{4-(x+3)^2}$ and $1=-\frac{2}{3}x+3$

(d) Find the total length (of the curve and the line) that is graphed above.

total length = length of semicircle + length of line

$=2\pi+\sqrt{25+16/9}$

semicircle:

$\frac{1}{2}[2\pi\cdot 2]=2\pi$

line: connects $(6,1)$ to $(1, 7/3)$

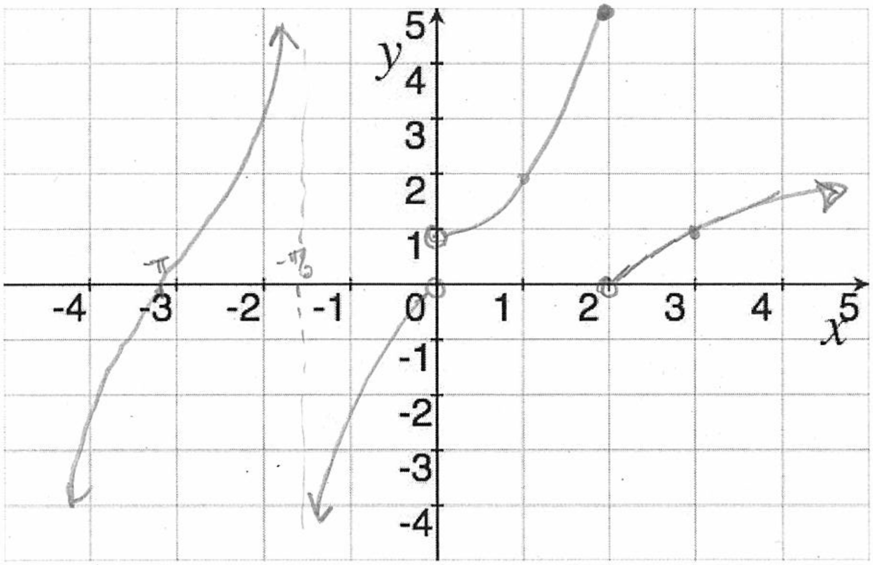
$\sqrt{5^2+(4/3)^2}$

5. Define f by

$$f(x) = \begin{cases} \tan x & \text{if } x < 0 \\ x^2 + 1 & \text{if } 0 < x \le 2 \\ \log_2(x-1) & \text{if } 2 < x \end{cases}$$

vert shift up 1

shift right 1 unit



(a) [8] Graph f on the axes above.

(b) [9] Find the following if possible:

$f(1)$

2 $\lceil 1^2+1 \rfloor$

$f(2) + f(3)$ $= 5 + 1 = 6$

$f(2) = 2^2 + 1 = 5$

$f(3) = \log_2(3-1) = \log_2(2) = 1$

$f(0)$

not defined

$f(\frac{-13\pi}{4})$ $= \tan(-13\pi/4)$

$= \tan(-13\pi/4 + \frac{8\pi}{4})$ b/c added a revolutio

$= \tan(-5\pi/4)$

$= \dfrac{\sin -5\pi/4}{\cos -5\pi/4}$

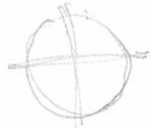
$= \dfrac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = -1$

Range of f

6. [8] Find all of the exact values x that satisfy the following:

$2\sin x = -\sqrt{3}$

$\sin x = \frac{-\sqrt{3}}{2}$



$x = -60°$ or coterminal angles with $-60°$

or

$x = -120°$ or coterminal angles with $-120°$

$5^{4x-1} = 7^x$

$\ln 5^{4x-1} = \ln 7^x$

$(4x-1)\ln 5 = x \ln 7$

$x4\ln 5 - \ln 5 = x\ln 7$

$x4\ln 5 - x\ln 7 = \ln 5$

$x(4\ln 5 - \ln 7) = \ln 5$

$x = \frac{\ln 5}{4\ln 5 - \ln 7}$

7. [4] Find all exact values for x that satisfy the following:

$\log(x-16) = 2 - \log(x-1)$

$\log(x-16) + \log(x-1) = 2$

$10^{\cancel{\log}(x-16)(x-1)} = 10^2$

$(x-16)(x-1) = 100$

$x^2 - 17x + 16 = 100$

$x^2 - 17x - 84 = 0$

$(x-21)(x+4) = 0$ domain problems - doesn't work in original problem

$\Rightarrow x = 21$ ~~or -4~~

$3^{5x}9^x = 27$

$3^{5x}(3^2)^x = 3^3$

$3^{5x}3^{2x} = 3^3$

$3^{5x+2x} = 3^3$

$\cancel{\log_3} 3^{5x+2x} = \cancel{\log_3} 3^3$

$5x + 2x = 3$

$7x = 3 \Rightarrow x = 3/7$

checks ✓

8. Simplify:

* Assume c, d + z are all greater than 0

$\frac{\sqrt{c^2d^6}}{\sqrt{4c^3d^{-4}}} = \frac{(c^2d^6)^{1/2}}{(4c^3d^{-4})^{1/2}}$

$= \frac{(c^2)^{1/2}(d^6)^{1/2}}{4^{1/2}(c^3)^{1/2}(d^{-4})^{1/2}} = \frac{|c|\,|d|^3}{2c^{3/2}d^{-2}}$

$= \frac{1}{2}|c|^{1-3/2}|d|^{3-2}$

$= \frac{1}{2}|c|^{1/2}|d|^5$

$2 - \log_5(25z)$

$2 - [\log_5 25 + \log_5 z]$

$2 - \log_5 5^2 - \log_5 z$

$2 - 2 - \log_5 z$

$-\log_5 z$

9. [7] Given $f(3) = 0$, use the connection between roots and factors to find the other roots of $f(x) = x^4 - 3x^3 - 25x^2 + 75x$

Since $f(3)=0$, 3 is a root
$\Rightarrow$ $x-3$ is a factor of $f(x)$

So

$$\begin{array}{r} x^3 - 25x \quad R\,0 \\ x-3 \,\big)\, x^4 - 3x^3 - 25x^2 + 75x \\ -(x^4 - 3x^3) \\ \hline -25x^2 + 75x \\ -(-25x^2 + 75x) \\ \hline 0 \end{array}$$

So $\dfrac{x^4 - 3x^3 - 25x^2 + 75x}{x-3} = x^3 - 25x$

$\Rightarrow x^4 - 3x^3 - 25x^2 + 75x = (x-3)(x^3 - 25x)$
$= (x-3)\,x\,(x^2 - 25)$
$= (x-3)\,x\,(x+5)(x-5)$

So $x-3$, x, $x+5$ + $x-5$ are factors

$\Rightarrow$ 3, 0, -5 + 5 are roots

10. Simplify:

$\sin^{-1}(\sin \frac{3\pi}{4}) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = ?$

$\Rightarrow \sin ? = 1/\sqrt{2}$ where
$-\pi/2 \leq ? \leq \pi/2$

$\Rightarrow ? = \pi/4$

$$\frac{\cos x}{1 - \sin x} + \frac{1 - \sin x}{\cos x}$$

$\dfrac{(\cos x)^2 + (1-\sin x)^2}{(\cos x)(1-\sin x)}$

$= \dfrac{\boxed{(\cos x)^2 + 1 - 2\sin x + (\sin x)^2}}{\cos x(1-\sin x)}$ Pyth. $= \dfrac{1 + 1 - 2\sin x}{\cos x(1-\sin x)}$

$= \dfrac{2 - 2\sin x}{\cos x(1-\sin x)} = \dfrac{2\cancel{(1-\sin x)}}{\cos x\cancel{(1-\sin x)}}$

$= \dfrac{2}{\cos x}$

11. [4] Let $\frac{-\pi}{2} < \theta < 0$ and $\cos\theta = \frac{1}{5}$. Find $\tan\theta$.

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ we need to find $\sin\theta$

Pyth: $(\sin\theta)^2 + (\cos\theta)^2 = 1$
$(\sin\theta)^2 + (\frac{1}{5})^2 = 1$
$(\sin\theta)^2 = 1 - 1/25 = 24/25$

$\sin\theta = \cancel{\pm}\dfrac{\sqrt{24}}{5}$
b/c sin is neg

$\Rightarrow \tan\theta = \dfrac{\frac{-\sqrt{24}}{5}}{\frac{1}{5}} = -\sqrt{24}$

12. [6] Let $\frac{\pi}{2} < \phi < \pi$ and $\frac{-\pi}{2} < \theta < 0$. Given that $\sin\phi = \frac{2}{3}$ and that $\cos\theta = \frac{1}{5}$, find $\cos(\theta + \phi)$ exactly. (You are free to use results from #10 above.)

$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$
$= 1/5 \cos\phi - \sin\theta \cdot 2/3$

from #11 $\sin\theta = -\dfrac{\sqrt{24}}{5}$ from the right

to find $\cos\phi$, use Pyth.
$(\cos\phi)^2 + (\sin\phi)^2 = 1$
$(\cos\phi)^2 = 1 - (2/3)^2 = 5/9$
$\cos\phi = \cancel{\pm}\dfrac{\sqrt{5}}{3}$
cos is neg

$\cos\phi = -\sqrt{5}/3$ so

$\cos(\theta+\phi) = \frac{1}{5} \cdot \frac{-\sqrt{5}}{3} - \frac{-\sqrt{24}}{5} \cdot \frac{2}{3} = \dfrac{-\sqrt{5} + 2\sqrt{24}}{15}$

12. [5] Your given a 16 oz mocha that is a rather weak 3% espresso. You, knowing you'll be up late studying mathematics, would rather like a 30% espresso drink. Realizing this you purchase an espresso machine. How much weak mocha do you discard and replace with straight espresso to have a 16 oz mocha with the desired concentration?

let x be the mocha you keep
y be the amount of espresso

$$\begin{array}{r} 16 \\ 3 \\ \hline 48 \end{array}$$

total mocha $\Rightarrow 16 = x + y$

total espresso $\Rightarrow .3(16) = .03x + y$

2 equations, 2 unknowns

$16 = x + y$ and $4.8 = .03x + y$

$\Rightarrow 16 - y = x$ sub into

to get $4.8 = .03(16 - y) + y$

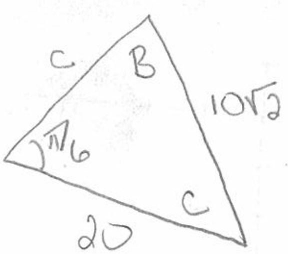
$4.8 = .48 - .03y + y$

$-.48 \quad -.48$

$4.32 = .97y$

$\Rightarrow y = \frac{4.32}{.97} = \frac{432}{97}$ oz

13. [5] Use the conventions from the book and class and let A be measure of the angle opposite the side with length a. Given that $a = 10\sqrt{2}$, $b = 20$, and $A = \frac{\pi}{6}$ with the standard notation, determine if the information describes 0, 1, or 2 triangles and solve for them/it if they/it exist/s.



$$\frac{\sin \pi/6}{10\sqrt{2}} = \frac{\sin B}{20}$$

$$\frac{20 \sin \pi/6}{10\sqrt{2}} = \sin B$$

$$\frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} \cdot \frac{1}{2} = \sin B$$

$\Rightarrow B = \pi/4$ or $3\pi/4$

Recall
$\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi$

$\boxed{\text{if } B = \pi/4}$

$C = \pi - \pi/4 - \pi/6$

$= \frac{12\pi - 3\pi - 2\pi}{12}$

$\boxed{\therefore C = 7\pi/12}$

$\Rightarrow \frac{\sin \pi/6}{10\sqrt{2}} = \frac{\sin \frac{7\pi}{12}}{c}$

$\frac{1}{20\sqrt{2}} = \frac{\sin\frac{7\pi}{12}}{c}$

$c = 20\sqrt{2} \sin \frac{7\pi}{12}$

$c = 20\sqrt{2} \sin\left(\frac{3\pi}{12} + \frac{4\pi}{12}\right)$

$c = 20\sqrt{2}\left[\sin \pi/4 \cos \pi/3 + \cos \pi/4 \sin \pi/3\right]$

$\boxed{c = 20\sqrt{2}\left[\frac{1}{\sqrt{2}}\frac{1}{2} + \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}\right] = 10(1+\sqrt{3})}$

$\boxed{\text{if } B = 3\pi/4}$

$C = \pi - 3\pi/4 - \pi/6$

$= \frac{12\pi - 9\pi - 2\pi}{12}$

$\boxed{\therefore C = \pi/12}$

$\frac{\sin \pi/6}{10\sqrt{2}} = \frac{\sin \pi/12}{c}$

$\frac{1}{20\sqrt{2}} = \frac{\sin \pi/12}{c}$

$c = 20\sqrt{2} \sin \pi/12$

$c = 20\sqrt{2} \sin\left(\frac{4\pi}{12} - \frac{3\pi}{12}\right)$

$c = 20\sqrt{2}\left[\sin \pi/3 \cos -\pi/4 + \cos \pi/3 \sin\right.$

$\boxed{c = 10(\sqrt{3} - 1)}$

14. [5] Suppose a radioactive isotope is such that one-fifth of the atoms in a sample decay after three years. Find the half-life of this isotope

use $P_0 2^{-t/h} = P(t)$

Start with P_0 + end with $\frac{4}{5}P_0$ when $t=3$.

$$\frac{\frac{4}{5}\cancel{P_0}}{\cancel{P_0}} = \frac{\cancel{P_0}\, 2^{-3/h}}{\cancel{P_0}}$$ solve for h.

$$\frac{4}{5} = 2^{-3/h}$$

$$\ln \frac{4}{5} = \frac{-3}{h} \ln 2$$

$$\frac{\ln \frac{4}{5}}{\ln 2} = \frac{-3}{h}$$

$$\frac{h \ln \frac{4}{5}}{\ln 2} = -3$$

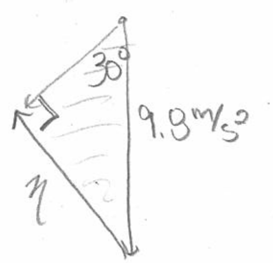
$$\Rightarrow h = \frac{-3 \ln 2}{\ln \frac{4}{5}}$$

15. [5] The force of friction is sometimes calculated by multiplying the normal force (the force holding the object up) by the mass of the object and by a 'coefficient of friction'. The coefficient of friction is a dimensionless number that depends on the two surfaces being pressed together.

A 10kg block is sliding down a dry glass ramp with angle of elevation of 60° and with a coefficient of friction of .94. Find the force of friction acting on the block.

need to find η & then compute $.94\eta$ (mass of obje

ie $.94\eta \cdot 10\text{kg}$



Sohcahtoa

$$\sin 30° = \frac{\eta}{9.8}$$

$$\Rightarrow \eta = 9.8 \sin 30° = 9.8 \cdot \tfrac{1}{2}$$

So the force of friction is $.94 \cdot 9.8 \cdot .5 \cdot 10 \ \text{m/s}^2 \cdot \text{kg}$