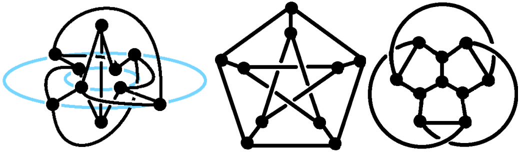
Topological Symmetry Groups Meet the Petersen Graph

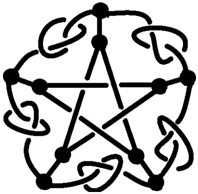
Ruth Vanderpool Ph. D.

School of Interdisciplinary Arts and Sciences
University of Washington, Tacoma

joint with Dr. Dwayne Chambers (UWT), Dr. Daniel Heath (PLU), Dr. Courtney Thatcher (UPS)



Outline



- Graph Automorphism Groups
- Examples & Motivation
- Topological Symmetry Groups (TSG)
- Examples and Results
- Petersen Graph meets TSG_+

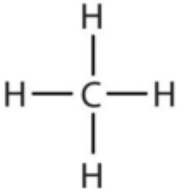
Automorphism Groups

Definition

An *automorphism* of a graph Γ is a permutation of the vertices that preserve adjacency. Let $Aut(\Gamma)$ denote the group of automorphisms Γ.

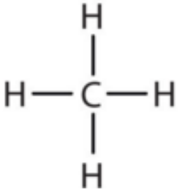
Definition

An *automorphism* of a graph Γ is a permutation of the vertices that preserve adjacency. Let $Aut(\Gamma)$ denote the group of automorphisms Γ.



Definition

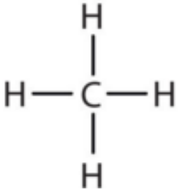
An *automorphism* of a graph Γ is a permutation of the vertices that preserve adjacency. Let $Aut(\Gamma)$ denote the group of automorphisms Γ.

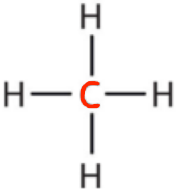


Automorphism Groups

Definition

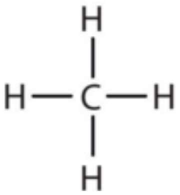
An *automorphism* of a graph Γ is a permutation of the vertices that preserve adjacency. Let $Aut(\Gamma)$ denote the group of automorphisms Γ.

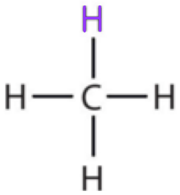




Definition

An *automorphism* of a graph Γ is a permutation of the vertices that preserve adjacency. Let $Aut(\Gamma)$ denote the group of automorphisms Γ.

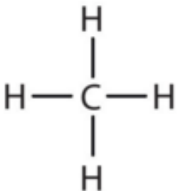


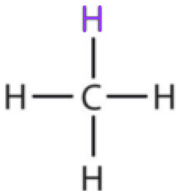


Automorphism Groups

Definition

An *automorphism* of a graph Γ is a permutation of the vertices that preserve adjacency. Let $Aut(\Gamma)$ denote the group of automorphisms Γ.



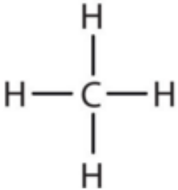


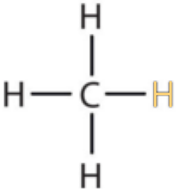
four choices

Automorphism Groups

Definition

An *automorphism* of a graph Γ is a permutation of the vertices that preserve adjacency. Let $Aut(\Gamma)$ denote the group of automorphisms Γ.

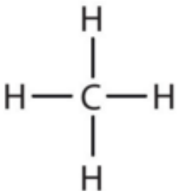


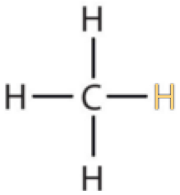


Automorphism Groups

Definition

An *automorphism* of a graph Γ is a permutation of the vertices that preserve adjacency. Let *Aut*(Γ) denote the group of automorphisms Γ.



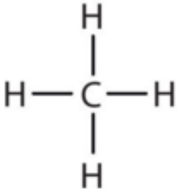


three choices

Automorphism Groups

Definition

An *automorphism* of a graph Γ is a permutation of the vertices that preserve adjacency. Let $Aut(\Gamma)$ denote the group of automorphisms Γ.

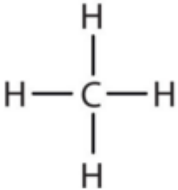


size of $Aut($  $)$: $4 * 3 * 2$

Automorphism Groups

Definition

An *automorphism* of a graph Γ is a permutation of the vertices that preserve adjacency. Let $Aut(\Gamma)$ denote the group of automorphisms Γ.

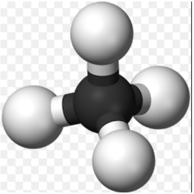


$Aut($  $) \cong S_4$

Automorphism Groups

Definition

An *automorphism* of a graph Γ is a permutation of the vertices that preserve adjacency. Let $Aut(\Gamma)$ denote the group of automorphisms Γ.

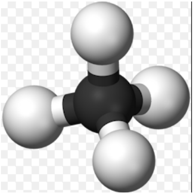


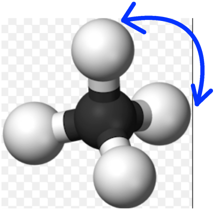
$Aut($  $) \cong S_4$

Automorphism Groups

Definition

An *automorphism* of a graph Γ is a permutation of the vertices that preserve adjacency. Let $Aut(\Gamma)$ denote the group of automorphisms Γ.





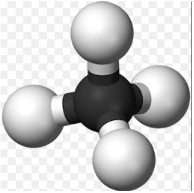
$Aut($  $) \cong S_4$

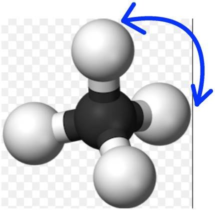
`http://symmetry.otterbein.edu/gallery/index.html`

Automorphism Groups

Definition

An *automorphism* of a graph Γ is a permutation of the vertices that preserve adjacency. Let $Aut(\Gamma)$ denote the group of automorphisms Γ.





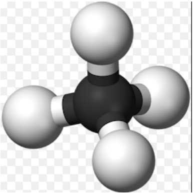
$Aut($  $) \cong S_4$

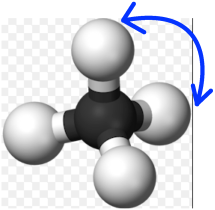
$Aut($  $) \neq$rigid movements in 3D

Automorphism Groups

Definition

An *automorphism* of a graph Γ is a permutation of the vertices that preserve adjacency. Let $Aut(\Gamma)$ denote the group of automorphisms Γ.





$Aut($  $) \cong S_4$

in 3D $\cong A_4$

Embeddings Matter!!!

A fix:

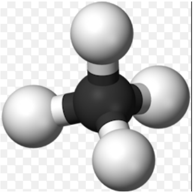
Definition

The *topological symmetry group* of a graph Γ embedded in S^3 is the subgroup of $Aut(\Gamma)$ induced by homeomorphisms of the graph in S^3. It is denoted by $TSG(\Gamma)$

A fix:

Definition

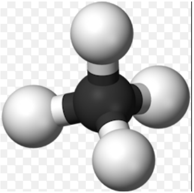
The *topological symmetry group* of a graph Γ embedded in S^3 is the subgroup of $Aut(\Gamma)$ induced by homeomorphisms of the graph in S^3. It is denoted by $TSG(\Gamma)$



A fix:

Definition

The *topological symmetry group* of a graph Γ embedded in S^3 is the subgroup of $Aut(\Gamma)$ induced by homeomorphisms of the graph in S^3. It is denoted by $TSG(\Gamma)$

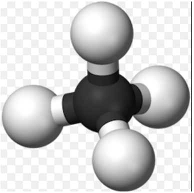


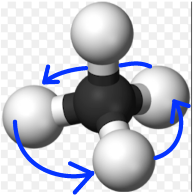
- Fix vertex and rotate opposite $\triangle$.

A fix:

Definition

The *topological symmetry group* of a graph Γ embedded in S^3 is the subgroup of $Aut(\Gamma)$ induced by homeomorphisms of the graph in S^3. It is denoted by $TSG(\Gamma)$



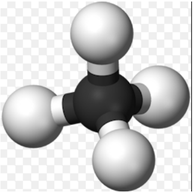


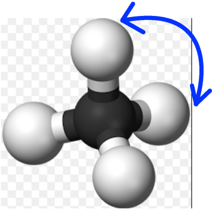
- Fix vertex and rotate opposite $\triangle$.

A fix:

Definition

The *topological symmetry group* of a graph Γ embedded in S^3 is the subgroup of $Aut(\Gamma)$ induced by homeomorphisms of the graph in S^3. It is denoted by $TSG(\Gamma)$





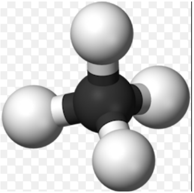
- Fix vertex and rotate opposite $\triangle$.
- Reflect over mirrors.

Topological Symmetry Group

A fix:

Definition

The *topological symmetry group* of a graph Γ embedded in S^3 is the subgroup of $Aut(\Gamma)$ induced by homeomorphisms of the graph in S^3. It is denoted by $TSG(\Gamma)$



- Fix vertex and rotate opposite $\triangle$.
- Reflect over mirrors.

$TSG($  $) \cong S_4$

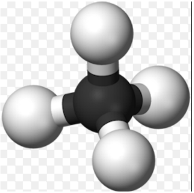
Recall we want $\cong A_4$

A better fix:

Definition

The orientation preserving topological symmetry group, $\mathrm{TSG}_+(\Gamma)$, is the subgroup of $\mathrm{TSG}(\Gamma)$ induced by orientation preserving homeomorphisms of (S^3, Γ).

Note: mirror symmetry is *not* included!

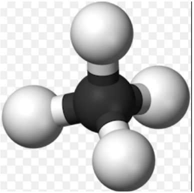


A better fix:

Definition

The orientation preserving topological symmetry group, $TSG_+(\Gamma)$, is the subgroup of $TSG(\Gamma)$ induced by orientation preserving homeomorphisms of (S^3, Γ).

Note: mirror symmetry is *not* included!



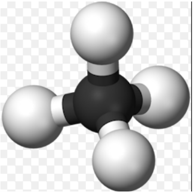
- Fix vertex and rotate opposite $\triangle$.

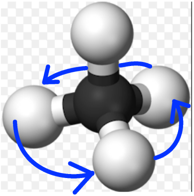
A better fix:

Definition

The orientation preserving topological symmetry group, $\mathrm{TSG}_+(\Gamma)$, is the subgroup of $\mathrm{TSG}(\Gamma)$ induced by orientation preserving homeomorphisms of (S^3, Γ).

Note: mirror symmetry is *not* included!





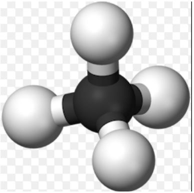
- Fix vertex and rotate opposite $\triangle$.

A better fix:

Definition

The orientation preserving topological symmetry group, $TSG_+(\Gamma)$, is the subgroup of $TSG(\Gamma)$ induced by orientation preserving homeomorphisms of (S^3, Γ).

Note: mirror symmetry is *not* included!



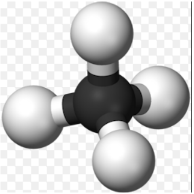
- Fix vertex and rotate opposite $\triangle$.
- ~~Reflect over mirrors.~~

A better fix:

Definition

The orientation preserving topological symmetry group, $\mathrm{TSG}_+(\Gamma)$, is the subgroup of $\mathrm{TSG}(\Gamma)$ induced by orientation preserving homeomorphisms of (S^3, Γ).

Note: mirror symmetry is *not* included!



- Fix vertex and rotate opposite $\triangle$.
- ~~Reflect over mirrors.~~

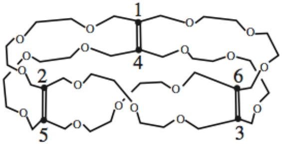
$TSG_+($  $) \cong A_4$

A better fix:

Definition

The orientation preserving topological symmetry group, $TSG_+(\Gamma)$, is the subgroup of $TSG(\Gamma)$ induced by orientation preserving homeomorphisms of (S^3, Γ).

Note: mirror symmetry is *not* included!

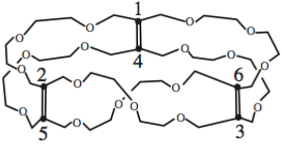


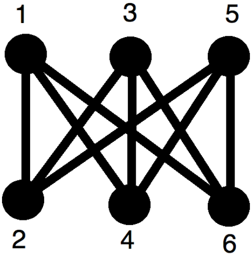
A better fix:

Definition

The orientation preserving topological symmetry group, $\mathrm{TSG}_+(\Gamma)$, is the subgroup of $\mathrm{TSG}(\Gamma)$ induced by orientation preserving homeomorphisms of (S^3, Γ).

Note: mirror symmetry is *not* included!



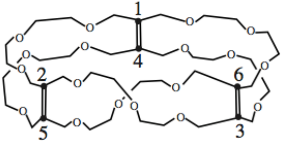


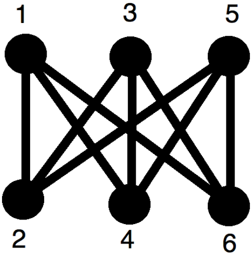
A better fix:

Definition

The orientation preserving topological symmetry group, $\mathrm{TSG}_+(\Gamma)$, is the subgroup of $\mathrm{TSG}(\Gamma)$ induced by orientation preserving homeomorphisms of (S^3, Γ).

Note: mirror symmetry is *not* included!





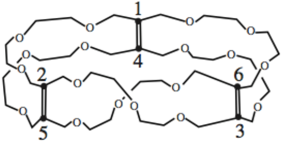
Aut() size 72

A better fix:

Definition

The orientation preserving topological symmetry group, $\mathrm{TSG}_+(\Gamma)$, is the subgroup of $\mathrm{TSG}(\Gamma)$ induced by orientation preserving homeomorphisms of (S^3, Γ).

Note: mirror symmetry is *not* included!



- Rigid motion: rotate upside down: $(2,3)(5,6)(1,4)$

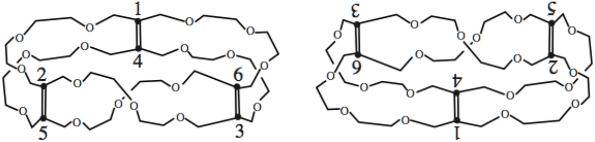
Aut() size 72

A better fix:

Definition

The orientation preserving topological symmetry group, $\mathrm{TSG}_+(\Gamma)$, is the subgroup of $\mathrm{TSG}(\Gamma)$ induced by orientation preserving homeomorphisms of (S^3, Γ).

Note: mirror symmetry is *not* included!



- Rigid motion: rotate upside down: $(2,3)(5,6)(1,4)$

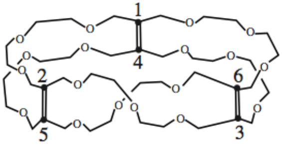
$Aut($$)$ size 72

A better fix:

Definition

The orientation preserving topological symmetry group, $\text{TSG}_+(\Gamma)$, is the subgroup of $\text{TSG}(\Gamma)$ induced by orientation preserving homeomorphisms of (S^3, Γ).

Note: mirror symmetry is *not* included!



- Rigid motion: rotate upside down: $(2,3)(5,6)(1,4)$
- Flexible: rotate 120° and “move the twist": (1,2,3,4,5,6)

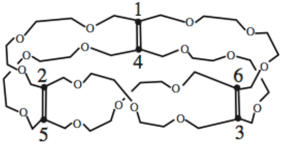
Aut() size 72

A better fix:

Definition

The orientation preserving topological symmetry group, $\mathrm{TSG}_+(\Gamma)$, is the subgroup of $\mathrm{TSG}(\Gamma)$ induced by orientation preserving homeomorphisms of (S^3, Γ).

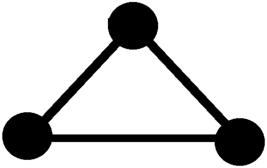
Note: mirror symmetry is *not* included!



- Rigid motion: rotate upside down: $(2,3)(5,6)(1,4)$
- Flexible: rotate 120° and "move the twist": (1,2,3,4,5,6)

$Aut($  $)$ size 72

$TSG_+($  $) \cong D_6$

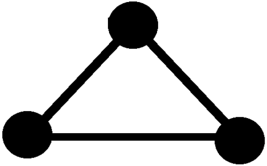


$Aut(\ ) \cong S_3$

$TSG(\ ) \cong$

$TSG_+(\ ) \cong$

Examples of TSG_+

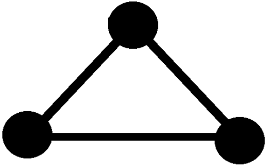


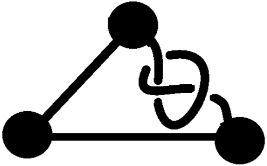
$Aut($$) \cong S_3$

$TSG($$) \cong S_3$

$TSG_+($$) \cong S_3$

Examples of TSG$_+$





$Aut(\;) \cong S_3$

$TSG(\;) \cong S_3$

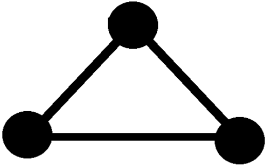
$TSG_+(\;) \cong S_3$

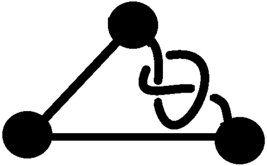
$Aut(\;) \cong S_3$

$TSG(\;) \cong$

$TSG_+(\;) \cong$

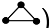
Examples of TSG_+





$Aut($$) \cong S_3$

$TSG($$) \cong S_3$

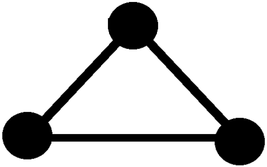
$TSG_+($$) \cong S_3$

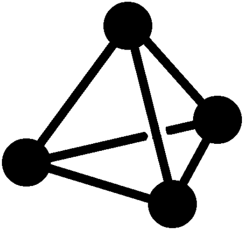
$Aut($$) \cong S_3$

$TSG($$) \cong S_3$

$TSG_+($$) \cong S_3$

Examples of TSG_+





$Aut(\ ) \cong S_3$

$TSG(\ ) \cong S_3$

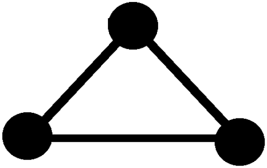
$TSG_+(\ ) \cong S_3$

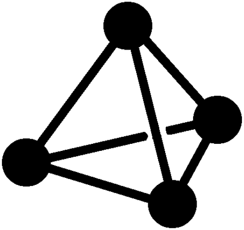
$Aut(\ ) \cong S_4$

$TSG(\ ) \cong$

$TSG_+(\ ) \cong$

Examples of TSG_+





$Aut(\ ) \cong S_3$

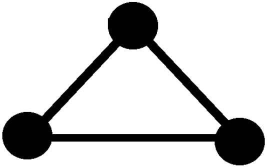
$TSG(\ ) \cong S_3$

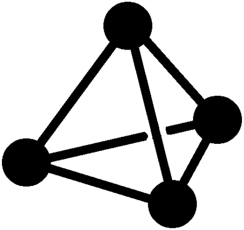
$TSG_+(\ ) \cong S_3$

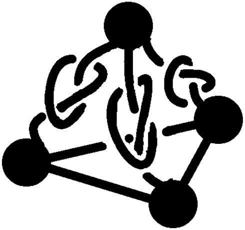
$Aut(\ ) \cong S_4$

$TSG(\ ) \cong S_4$

$TSG_+(\ ) \cong A_4$







$Aut(\;) \cong S_3$

$TSG(\;) \cong S_3$

$TSG_+(\;) \cong S_3$

$Aut(\;) \cong S_4$

$TSG(\;) \cong S_4$

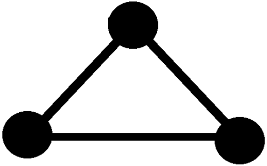
$TSG_+(\;) \cong A_4$

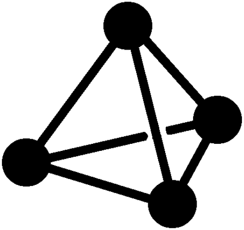
$Aut(\;) \cong S_4$

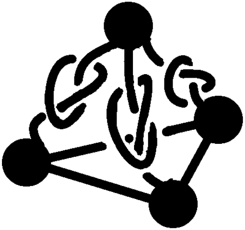
$TSG(\;) \cong$

$TSG_+(\;) \cong$

Examples of TSG_+







$Aut($$) \cong S_3$

$TSG($$) \cong S_3$

$TSG_+($$) \cong S_3$

$Aut($$) \cong S_4$

$TSG($$) \cong S_4$

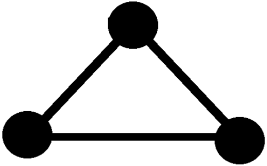
$TSG_+($$) \cong A_4$

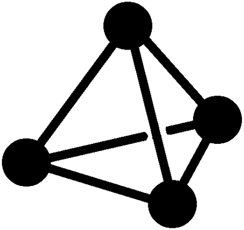
$Aut($$) \cong S_4$

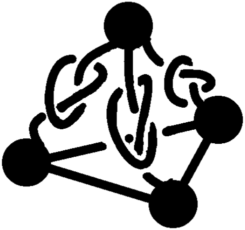
$TSG($$) \cong \mathbb{Z}_3$

$TSG_+($$) \cong \mathbb{Z}_3$

Examples of TSG_+







$Aut($$) \cong S_3$

$TSG($$) \cong S_3$

$TSG_+($$) \cong S_3$

$Aut($$) \cong S_4$

$TSG($$) \cong S_4$

$TSG_+($$) \cong A_4$

$Aut($$) \cong S_4$

$TSG($$) \cong \mathbb{Z}_3$

$TSG_+($$) \cong \mathbb{Z}_3$

Each embedding can give different groups....

Examples of TSG$_+$

Graph	Polyhedral Groups	$\mathbb{Z}_m$ and D_m	$\mathbb{Z}_r \times \mathbb{Z}_s$ and $(\mathbb{Z}_r \times \mathbb{Z}_s) \rtimes \mathbb{Z}_2$	$\mathbb{Z}_r \times D_s$ and $D_r \times D_s$
K_2	None	$\mathbb{Z}_2$	None	None
K_3	None	$\mathbb{Z}_3$, D_3	None	None
K_4	A_4, S_4	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, D_2, D_3, D_4	None	None
K_5	A_4, A_5	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$, D_2, D_3, D_5	None	None
K_6	None	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$, $\mathbb{Z}_6$, D_2, D_3, D_5, D_6	$\mathbb{Z}_3 \times \mathbb{Z}_3$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	$\mathbb{Z}_3 \times D_3$, $D_3 \times D_3$
K_7	None	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$, $\mathbb{Z}_7$, D_3, D_5, D_7	None	None
K_8	A_4, S_4	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_7$, $\mathbb{Z}_8$, D_2, D_3, D_4, D_5, D_7, D_8	None	None
K_9	None	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_7$, $\mathbb{Z}_9$, D_2, D_3, D_7, D_9	$\mathbb{Z}_3 \times \mathbb{Z}_3$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	None
K_{10}	None	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$, $\mathbb{Z}_7$, $\mathbb{Z}_9$, $\mathbb{Z}_{10}$, D_2, D_3, D_5, D_7, D_9, D_{10}	None	None
K_{11}	None	$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$, $\mathbb{Z}_9$, $\mathbb{Z}_{11}$, D_3, D_5, D_9, D_{11}	None	None

(Flapan, Mellor, Naimi, Yoshizawa 2013)

If Γ is an embedding in S^3 of a graph γ,
what $TSG_+(\Gamma)$ are possible?

- Classified possible groups of TSG_+ for complete graphs. (Flapan, Mellor, Naimi, Yoshizawa 2013)

If Γ is an embedding in S^3 of a graph γ,
what $TSG_+(\Gamma)$ are possible?

- Classified possible groups of TSG_+ for complete graphs. (Flapan, Mellor, Naimi, Yoshizawa 2013)
- $TSG_+(\Gamma) \leq Aut(\gamma)$

TSG_+ Possibilities

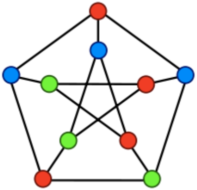
If Γ is an embedding in S^3 of a graph γ,
what $TSG_+(\Gamma)$ are possible?

- Classified possible groups of TSG_+ for complete graphs. (Flapan, Mellor, Naimi, Yoshizawa 2013)
- $TSG_+(\Gamma) \leq Aut(\gamma)$

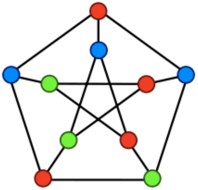
Complete Graph Theorem (Flapan, Naimi and Tamvakis 2006)

A finite group H is $TSG_+(\Gamma)$ for some embedding Γ of a complete graph in S^3 if and only if H is isomorphic to a finite cyclic group, a dihedral group, A_4, S_4, A_5, or a finite subgroup of $D_m \times D_m$ for some odd m.

Petersen Graph

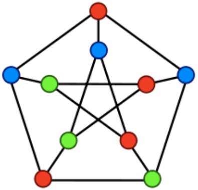


Petersen Graph



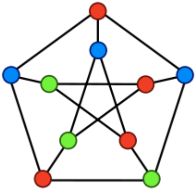
- not complete but 3-connected

Petersen Graph



- not complete but 3-connected
- “a remarkable configuration that serves as a counterexample to many optimistic predications about what might be true of graphs in general" -Donald Knuth

Petersen Graph



- not complete but 3-connected
- “a remarkable configuration that serves as a counterexample to many optimistic predications about what might be true of graphs in general" -Donald Knuth
- Inspired books!

Petersen Graph meets TSG

Plan:

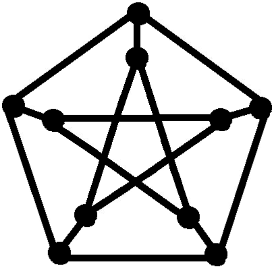
1. Find $Aut($  $)$ and all the subgroups.

   (Recall TSG_+ is a subset of $Aut($  $))$

2. For each subgroup H of $Aut($  $)$, either
   - show there exists no embedding Γ with $TSG_+(\Gamma) \cong H$
   - provide an embedding Γ where $TSG_+(\Gamma) \cong H$

$Aut($  $) \cong S_5$

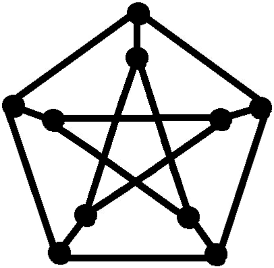
Step 1)



$Aut($  $)$

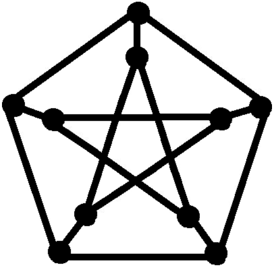
$Aut($  $) \cong S_5$

Step 1)



$Aut($  $)$ ... there are 10 vertices & not complete

Step 1)

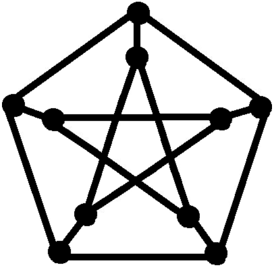


Aut()

- Label each vertex with two numbers from $\{1, 2, 3, 4, 5\}$

$Aut($  $) \cong S_5$

Step 1)

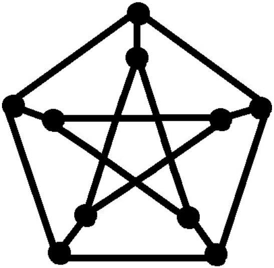


$Aut($  $)$

- Label each vertex with two numbers from $\{1, 2, 3, 4, 5\}$
  $(1, 2)$, $(1, 3)$, $(1, 4)$, $(1, 5)$, $(2, 3)$,
  $(2, 4)$, $(2, 5)$, $(3, 4)$, $(3, 5)$, $(4, 5)$

$Aut($  $) \cong S_5$

Step 1)

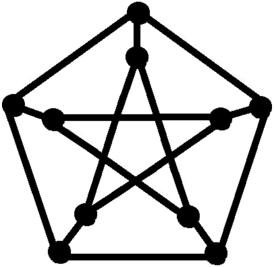


$Aut($  $)$

- Label each vertex with two numbers from $\{1, 2, 3, 4, 5\}$
- An edge exists between two vertices if the intersection between their two labels is empty.

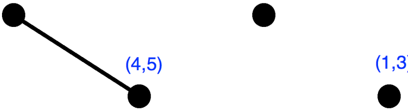
$Aut($  $) \cong S_5$

Step 1)



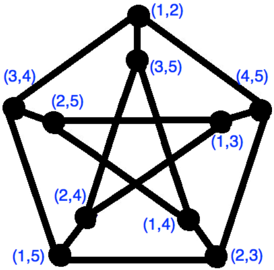
$Aut($  $)$

- Label each vertex with two numbers from $\{1, 2, 3, 4, 5\}$
- An edge exists between two vertices if the intersection between their two labels is empty.



$Aut($  $) \cong S_5$

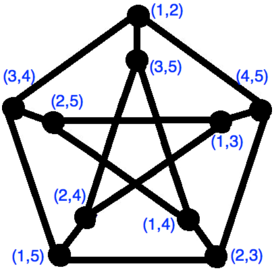
Step 1)



$Aut($  $)$

- Label each vertex with two numbers from $\{1, 2, 3, 4, 5\}$
- An edge exists between two vertices if the intersection between their two labels is empty.
- Automorphisms are determined by permuting the numbers $\{1, 2, 3, 4, 5\}$

Step 1)

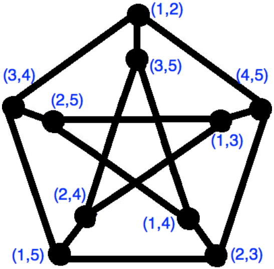


$Aut($  $)$

- Label each vertex with two numbers from $\{1, 2, 3, 4, 5\}$
- An edge exists between two vertices if the intersection between their two labels is empty.
- Automorphisms are determined by permuting the numbers $\{1, 2, 3, 4, 5\}$
  Consider $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 1$

$Aut($  $) \cong S_5$

Step 1)



$Aut($  $)$

- Label each vertex with two numbers from $\{1, 2, 3, 4, 5\}$
- An edge exists between two vertices if the intersection between their two labels is empty.
- Automorphisms are determined by permuting the numbers $\{1, 2, 3, 4, 5\}$
  Consider $1 \to 2 \to 3 \to 4 \to 5 \to 1$
  $(1, 2) \longrightarrow (2, 3)$ $(2, 5) \longrightarrow (1, 3)$

Subgroups of *Aut*()

Step 2)

Subgroups of *Aut*():

Subgroups of *Aut*()

Step 2)

Subgroups of *Aut*():

S_5	A_5
S_4	A_4
D_6	D_5
D_4	D_3
$\mathbb{Z}_6$	$\mathbb{Z}_5$
$\mathbb{Z}_4$	$\mathbb{Z}_3$
$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$S_3 \times \mathbb{Z}_2$	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$

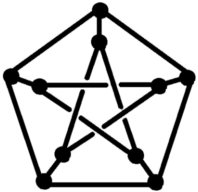
Subgroups of $Aut($  $)$

Step 2)

Subgroups of $Aut($  $)$:

S_5
S_4
D_6
D_4
$\mathbb{Z}_6$
$\mathbb{Z}_4$
$\mathbb{Z}_2$
$S_3 \times \mathbb{Z}_2$

A_5
A_4
D_5
D_3
$\mathbb{Z}_5$
$\mathbb{Z}_3$
$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$

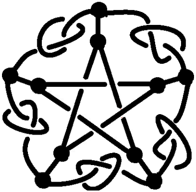


Subgroups of *Aut*()

Step 2)

Subgroups of *Aut*():

S_5	A_5
S_4	A_4
D_6	D_5
D_4	D_3
$\mathbb{Z}_6$	$\mathbb{Z}_5$
$\mathbb{Z}_4$	$\mathbb{Z}_3$
$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$S_3 \times \mathbb{Z}_2$	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$



Step 2)

Subgroups of *Aut*():

S_5
S_4
D_6
D_4
$\mathbb{Z}_6$
$\mathbb{Z}_4$
$\mathbb{Z}_2$
$S_3 \times \mathbb{Z}_2$

A_5
A_4
D_5
D_3
$\mathbb{Z}_5$
$\mathbb{Z}_3$
$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$

References

- http://symmetry.otterbein.edu/gallery/index.html
- E. Flapan, R. Naimi, H. Tamvakis, Topological Symmetry Groups of Complete Graphs in the 3-Sphere, Journal of the London Mathematical Society, 73, (2006), 237-251.
- E. Flapan, B. Mellor, R. Naimi, M. Yoshizawa. Classification of Topological Symmetry Groups of K_n, arXive: http://arxiv.org/pdf/1205.1560.pdf
- D.A. Holton, J. Sheehan. The Petersen Graph, Cambridge University Press, (1993)