Markowitz Portfolio Analysis for the Individual Investor

by

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INTRODUCTION

The portfolio is a kind of hedge against the unforseen. A large number of stocks, bonds, etc., put together into a portfolio can protect the investor against the inherently unpredictable fluctuations by offsetting those that decrease in value against those that increase. For this to work it is important that the portfolio be diverse, that is, contain many types of investments, to assure that there will be plenty of gainers to offset the losers.

But how to diversify? One strategy would be to randomly select from a large universe of investments. Without any prior knowledge about the behavior of the investment, we might expect gains and losses to be random. If the universe were defined as securities included in an index, the S&P 500, for example, we might expect to do as well as that index. A study published in 1964 by Lawrence Fisher and James Lorie ("Rates of Return on Investments in Common Stocks", Journal of Business, 37:1-24) showed that the "dart board" portfolio did as well as any alternative investment strategy. Since there are mutual funds that do not perform as well as the S&P 500, however, many investors should perhaps consider the random portfolio.

The large funds can do (usually) better than random because they have the resources to diversify very broadly. Is there a way for the individual investor with limited resources to select a portfolio offering the kind of protection available to the large portfolios? Yes, there is. As you will see below, by applying a mathematical technique heretofore only available to the big funds called the Markowitz asset allocation method, it is possible to find an optimal portfolio that will do better, in many cases far better, than a random portfolio. This method can be used both strategically, to inform decisions about distributions of assets into markets, as well as tactically, in setting allocations that improve the money-making power of your personal portfolio.

RISK AND RETURN

Harry M. Markowitz won the Nobel Prize in 1990 for contributions to the science of portfolio management which included the observation that risk must be taken into account as well as return. Risk of an individual investment is measured as its *variance*, i.e., its volatility, the degree of its fluctuations about its mean. Intuitively this makes sense because high fluctuations mean greater unpredictability and therefore greater risk.

The risk of a portfolio of investments is more complicated because the correlations among the investments must be considered. If two stocks, for example, are highly negatively correlated, the net risk of a portfolio of these stocks would be zero, i.e., gains in one would completely offset the losses in the other. Two positively correlated stocks, on the other hand, are indistinguishable to the degree that they are correlated, and thus their fluctuations add to risk rather than subtract.

The essence of portfolio analysis is the balance of return against risk. These concepts must be defined mathematically and combined into a *utility* function. The utility function also contains the asset allocation as an unknown. The Markowitz method finds the asset allocation that maximizes the rate of return given the measure of risk. Before we can proceed, however, we need data or information on the measures of risk and rates of return.

The Data. For our analysis we require the following quantities:

- *mean rates of return*, i.e., some estimate of the the average rate of return of each security in the portfolio.
- the *standard deviation* of rates of return. This is square root of the variance of the rates.
- the *correlation* of each rate of return with each of the others.
- *risk tolerance*, a fixed value set by the investor to reflect the portfolio tolerance for risk.
- *risk free rate of return*, usually the rate of return for a T-bill. We will use this to determine the minimum variance portfolio.

In most circumstances, these quantities will be computed from historical data over a selected time period. They may also be estimated from experience or taken from studies found in the finance literature. The time period can be a month, quarter, or year. The number of time points must exceed the number of securities included and thus determines the maximum size of portfolio that can be analyzed.

In addition, the following may be specified in more advanced analysis,

- *allocation constraints*, you may choose to constrain the allocation proportions to specified intervals.
- *sector constraints*, you may divide the securities into sectors and then constrain the total allocations within the sectors.

The Allocation Vector – what we seek. The portfolio is described as weighted function of its individual elements. These weights are simple proportions, and set of proportions together is called the *allocation vector*. If our portfolio was composed of 30% of one stock and the rest of another, the weights would be .3 and .7 respectively. Each stock, therefore, has a weight which is its proportion of the total portfolio. The allocation vector is unknown at first, and the determination of this vector is the ultimate goal of this analysis.

Portfolio Return. The portfolio return is the average rate return of the individual security rates weighted by how much each security participates in the portfolio, i.e., by the allocation vector.

Portfolio Risk. The portfolio risk is a function of the standard deviations of the individual rates and of their correlations with each other weighted by the allocation vector.

Risk Tolerance. The risk tolerance is a measure of the willingness of the investor to sustain risk. We shall see in the Advanced Topics section that it is not necessary to establish your risk tolerance very closely. As a rule of thumb, the value selected should be roughly equal to 100 minus the percent of the asset exposure in this portfolio. Thus if, say, 90% of your assets are to be invested in this portfolio then you would choose a risk tolerance of 10, or if, say, 10% of your assets were being invested, it would be 90.

MATHEMATICAL IDEAS.

Here is some basic math:

Mean Returns. The individual mean rate of return of each security over the selected time period is

$$\mu_i = \sum r_{it} / N$$

where r_{it} is the rate of return of the i-th security at time t. For example, suppose x_{it} represent a stock's value at time t. Then

$$r_{it} = \frac{x_{it} - x_{i0}}{x_{i0}}$$

Usually the rate is *annualized*, i.e., compounded to a yearly rate. Thus, if r_{it} is a monthly rate,

$$ar_{it} = (1 + r_{it})^{12} - 1$$

Finally, multiply by 100 to get a percentage.

An alternative to annualization is to scale the risk tolerance by the ratio of the time period of the rate to a year. When using a monthly rate, for example, divide the risk tolerance by 12.

Standard Deviations. The standard deviation is a bit more mathematically complicated:

$$\sigma_i = \sqrt{\sum (r_{ti} - \mu_i)^2 / N}$$

Correlations. The correlation of the i-th security with the j-th security is

$$\rho_{ij} = \frac{1}{\sigma_i \sigma_j} \sqrt{\sum (r_{ti} - \mu_i)(r_{tj} - \mu_j) / N}$$

Portfolio Return. The portfolio return is the average of the individual mean returns weighted by the allocation vector. Thus if the mean of the return of the first stock in a theoretical portfolio over the time period is 14.52 and that of the second is12.3, then the total mean return is (.3)(14.52) + (.7)(12.3) = 12.97.

The portfolio return, then, is

$$\mu_{total} = \sum a_i \mu_i$$

where μ_{total} is the total return, and a_i is the allocation or proportion of the i-th security in the portfolio.

Portfolio Risk. The portfolio risk is a function of the correlations and standard deviations weighted by the allocation weights,

$$\sigma_{total} = \sqrt{\sum a_i^2 \sigma_i^2 + 2\sum a_i a_j \sigma_i \sigma_j \rho_{ij}}$$

Utility Function. The utility function is

$$U = (T\mu_{total} - \sigma_{total}^2) / 2$$

Where T is the risk tolerance. The known quantities in this function are the means, standard deviations, correlations of the securities, and the risk tolerance. The unknown quantities to be determined are the allocation weights, an a for each security. A computer program is used to find values for each a_i such that U is maximum.

AN OPTIMAL PORTFOLIO

To keep the example below simple only six stocks are included in the portfolio. Historical data on these six stocks were collected from a one year period, and means, standard deviations and correlations were computed.

	Means	Standard Deviations
Merrill Lynch	15.852	37.215
Paine Webber	14.262	41.773
Digital Equipment	31.336	33.165
Microsoft	25.775	62.009
Silicon Graphics	50.228	60.720
Hewlett-Packard	14.842	23.757

Correlations

Merrill Lynch	1.000	0.944	0.146	0.231	0.379	0.258
Paine Webber	0.944	1.000	0.109	0.239	0.413	0.223
Digital Equip	0.146	0.109	1.000	-0.169	-0.229	0.691
Microsoft	0.231	0.239	-0.169	1.000	0.882	-0.256
Silicon Graphics	0.379	0.413	-0.229	0.882	1.000	-0.284
Hewlett-Packard	0.258	0.223	0.691	-0.256	-0.284	1.000

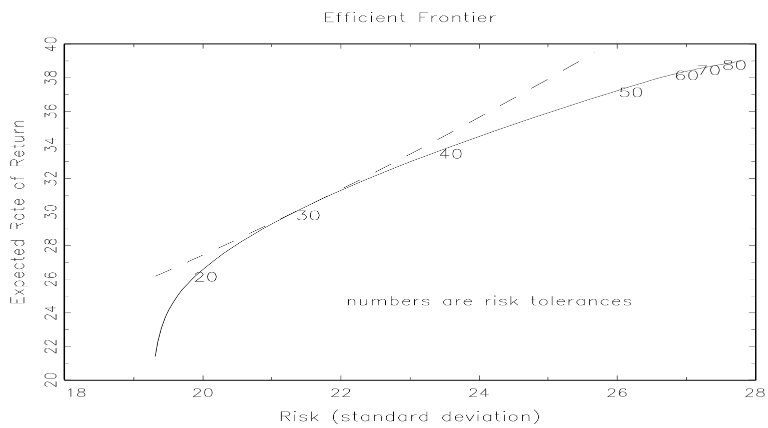
An allocation vector was computed for each of a range of risk tolerances:

risk tolerance	20	30	40	50	60	70
portfolio returns	29.73	33.40	37.06	38.07	38.38	38.69
portfolio risk	21.20	23.26	25.87	26.68	27.00	27.37
Merrill Lynch	----	----	----	----	----	----
Paine Webber	----	----	----	----	----	----
Digital Equipment	0.34	0.48	0.63	0.64	0.63	0.61
Microsoft	----	----	----	----	----	----
Silicon Graphics	0.26	0.30	0.34	0.36	0.37	0.39
Hewlett-Packard	0.40	0.22	0.04	----	----	----

As the risk tolerance increases, the risk, as measured by the portfolio standard deviation, increases. And in accord with what we should expect, as the risk increases, the portfolio return increases. Interestingly, as we become more risk tolerant, the portfolio, as indicated by changes in the allocation proportions, shifts from Hewlett-Packard to Paine Webber and Microsoft, and the increase in the shift is greater with Paine Webber than it is with Microsoft. This analysis suggests that a conservative investor would select more Hewlett-Packard stock, while an assertive investor would do well to concentrate on Paine Webber and Microsoft.

Efficient Frontier. The plot of the portfolio returns against standard deviations over the range of risk tolerances is called the efficient frontier.

The dashed line in the plot is a straight line tangent to the curved line relating portfolio return to portfolio risk, that passes through the line of zero risk at the rate of return for a *riskless* asset, in this case the return for a T-bill. The point at which the line touches the curve is the portfolio with the highest rate of return with the minimum variance. It is sometimes called the minimum variance portfolio, and it is our optimal portfolio.

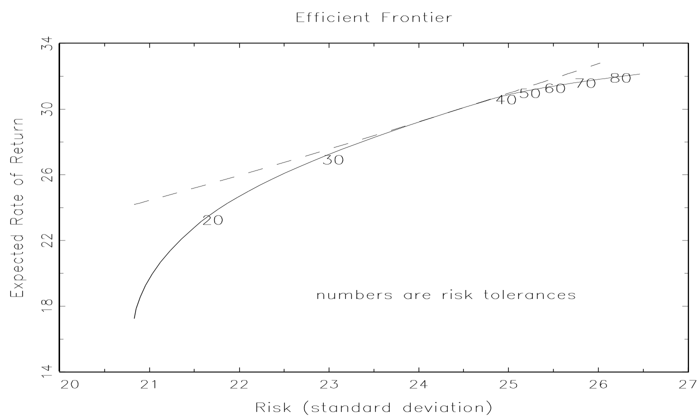


The line touches the curve at about risk tolerance = 30. This implies that the minimum variance portfolio would be 48% Digital Equipment, 30% Silicon Graphics, and 22% Hewlett-Packard.

Sector Constraints. The results of the previous analysis ended up putting all of the portfolio into the computer related companies. This lack of diversification can be avoided by entering sector constraints into the model. First, the stocks are grouped into sectors, and then maximum and/or minimum allocations are specified for each sector. Define two sectors, finance and computer, and let us constrain the allocation to the representation of the sectors in the portfolio, i.e., 1/3rd finance and 2/3rds computer. The sector constrained results are:

risk tolerance	20	30	40	50	60	70
portfolio returns	26.83	30.50	30.89	31.20	31.51	31.82
portfolio risk	22.82	24.74	25.01	25.29	25.63	26.02
Merrill Lynch	0.30	0.30	0.30	0.30	0.30	0.30
Paine Webber	----	----	----	----	----	----
Digital Equipment	0.35	0.49	0.48	0.46	0.44	0.43
Microsoft	----	----	----	----	----	----
Silicon Graphics	0.17	0.21	0.22	0.24	0.26	0.27
Hewlett Packard	0.19	0.00	----	----	----	----

and the efficient frontier:



The minimum variance portfolio is at about risk tolerance = 38. At that risk tolerance Digital Equipment is untouched at 48%, Silicon Graphics is reduced somewhat to 22%, and Merrill Lynch replaces Hewlett-Packard with 30% of the portfolio.

A Tactical Analysis

Usually, the large investment funds apply the Markowitz asset allocation analysis to data collected over relatively large time periods, monthly rates over 5 years say. Shorter time periods, however, are also possible. For illustration, daily returns are analyzed for 21 stocks over a 90 day period. A hypothetical $10,000 portfolio is allocated accorded to this analysis. Then in 30 day segments the stocks are again analyzed and the portfolio re-allocated according to the new results.

In the following analysis a risk tolerance of 60 (scaled to a daily value by dividing by 365) is used. Of the 21 stocks, only the seven presented in the table were selected for allocation.

days	CPL	DEC	DIS	GFC	MHC	OEC	CNNR	MSFT	SGIC	Portfolio
90	----	0.68	----	----	0.25	0.06	----	----	----	10000.00
120	----	----	0.45	----	0.55	----	----	----	----	9340.34
150	----	----	0.71	----	0.11	----	0.19	----	----	10055.78
180	----	----	0.64	----	0.22	----	0.14	----	----	10552.14
210	----	----	0.82	0.01	0.14	----	0.03	----	----	11865.92
240	----	----	0.72	----	0.14	----	0.09	----	0.05	12081.10
260	0.12	----	0.77	----	0.01	----	0.01	0.07	0.03	12264.50

The 22.64% rate of return of this portfolio is 33.18% when computed at an annual rate. If the $10,000 had been invested equally in the 21 stocks, it would have been valued at $11,001.20 after the 260 days, which represents an annualized rate of return of 14.43%. The tactical use of the Markowitz asset allocation method in this case represents a gain of more than a thousand dollars in less than 9 months over a naive allocation.

Transaction costs are not taken into account here. It should be noted that the naive portfolio involves one transaction on 21 stocks while the Markowitz method portfolio involves 6 transactions on 7 stocks.

It should also be noted that the stocks used in this analysis were arbitrarily selected. Better results should be expected from a larger base of securities from a mix of markets.

A STRATEGIC ANALYSIS

To this point, the discussion has centered on the optimal selection of individual investments in a portfolio. Of equal or greater importance, however, is the determination of an overall investment strategy involving the optimal allocation of assets across *markets*. The Markowitz mean/variance method also has application to this type of problem. In "The Emerging Markets and Strategic Asset Allocation", Douglas Stone analyzed the role of emerging markets, those in Latin America, Southeast Asia, for example (The Journal of Investing, Summer, 1992, pages 40 to 45). He showed there that the efficient frontier for a portfolio that included the emerging markets was greater at any risk tolerance than one without.

The study used the following data calculated from historical data over the 1979 to 1990 time period:

	means	standard deviations
Russell 1000 Index	12.5	17.0
MSCI EAFE Index	12.5	19.0
91-day T-bill	7.5	1.0
Gov/Corp. Bonds	8.8	5.5
Emerging Markets	12.5	23.0
US Equity, Small Cap	14.0	21.0
Real Estate	10.0	13.0

correlation matrix

Russell 1000 Ind.	1.00	0.47	0.07	0.33	0.23	0.91	0.10
MSCI EAFE Index	0.47	1.00	0.19	0.24	0.24	0.43	0.00
91-day T-bill	0.07	0.19	1.00	0.09	0.17	0.03	0.25
Gov/Corp. Bonds	0.33	0.24	0.09	1.00	0.11	0.24	0.00
Emerging Markets	0.23	0.24	0.17	0.11	1.00	0.21	0.00
US Equity, Sm.Cap	0.91	0.43	0.03	0.24	0.21	1.00	0.10
Real Estate	0.10	0.00	0.25	0.00	0.00	0.10	1.00

Stone imposed the following constraints on the allocation vector:

	minimum	maximum
Russell 1000 Index	0.27	0.58
MSCI EAFE Index	0.00	0.15
91-day T-bill	0.00	0.24
Gov/Corp. Bonds	0.12	0.72
Emerging Markets	0.00	1.00
US Equity, Small Cap	0.00	1.00
Real Estate	0.00	0.14

The constraints on allocations for the Russell 1000 Index, the MSCI EAFE Index, the T-bills, the bonds, and real estate were based on U.S. client asset allocations for the Russell Frank company.

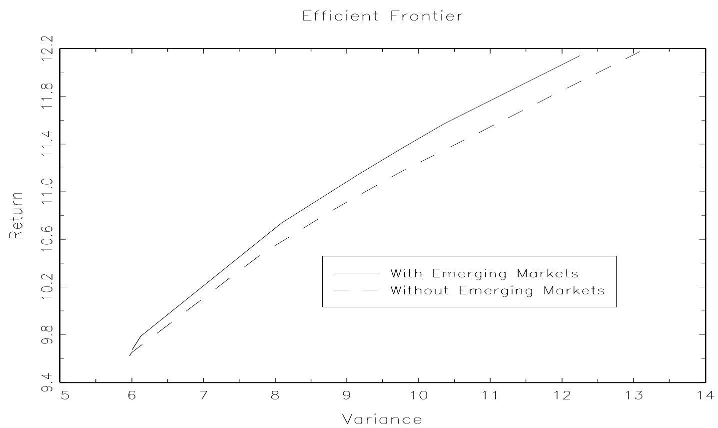
The results are:

Risk Tolerance	10	17	40	50	55	60	100
Russell 1000 Ind.	0.27	0.27	0.27	0.27	0.27	0.27	0.27
MSCI EAFE Index	----	----	0.10	0.13	0.14	0.15	0.15
91-day T-bill	0.24	0.24	----	----	----	----	----
Gov/Corp. Bonds	0.36	0.32	0.38	0.29	0.24	0.19	0.12
Emerging Markets	0.01	0.04	0.11	0.13	0.14	0.15	0.17
US Equity, Sm.Cap	----	----	----	0.04	0.07	0.09	0.20
Real Estate	0.11	0.13	0.14	0.14	0.14	0.14	0.09
Return	9.68	9.79	10.74	11.15	11.36	11.57	12.14
Risk	6.00	6.13	8.09	9.16	9.76	10.35	12.26

When the Emerging Markets are removed from the analysis the results become,

Risk Tolerance	10	17	40	50	55	60	100
Russell 1000 Ind.	0.27	0.27	0.27	0.27	0.27	0.27	0.27
MSCI EAFE Index	----	----	0.13	0.15	0.15	0.15	0.15
91-day T-bill	0.24	0.24	----	----	----	----	----
Gov/Corp Bonds	0.38	0.36	0.46	0.38	0.35	0.32	0.12
US Equity, Sm.Cap	----	----	0.01	0.06	0.09	0.12	0.32
Real Estate	0.11	0.13	0.14	0.14	0.14	0.14	0.14
Return	9.62	9.65	10.48	10.85	11.01	11.17	12.19
Risk	5.97	6.00	7.81	8.81	9.28	9.77	13.12

The efficient frontier plot for these two analyses is



As you can see, at every level of risk, i.e., at a given portfolio standard deviation, the return for the portfolio including the emerging markets is higher. We conclude that investors would improve their portfolios by including securities from emerging markets.

CONCLUSION

Both strategically and tacticly, the Markowitz asset allocation analysis provides investors with information to optimally distribute their assets. For the most protection against the unknown, assets should be broadly diversified.

The first step is a "strategic" analysis of markets – how assets should be distributed among stocks, bonds, real estate, European markets, merging markets, commodities, and so on. Then, once the markets have been decided upon, the investor can perform additional, "tactical" analyses for the optimal allocation within those markets.

Ideally, these analyses can be repeated at regular periods, and the allocation adjusted accordingly, thus assuring a long term optimal distribution of one's financial resources.

RECOMMENDED READING

Capital Ideas by Peter L. Bernstein. The Free Press, 1992.

A Random Walk Down Wall Street by Burton G. Malkiel. W.W. Norton, 1990.

Modern Investment Theory, 3rd Ed. by Robert A. Haugen. Simon & Schuster, 1993.

Quantitative Methods for Financial Analysis edited by Stephen J. Brown and Mark P. Kritzman. Dow Jones-Irwin, 1987.