STABILITY LIMITS IN RESONANT PLANETARY SYSTEMS

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ABSTRACT

The relationship between the boundaries for Hill and Lagrange stability in orbital element space is modified in the case of resonantly interacting planets. Hill stability requires the ordering of the planets to remain constant, while Lagrange stability also requires all planets to remain bound to the central star. The Hill stability boundary is defined analytically, but no equations exist to define the Lagrange boundary, so we perform numerical experiments to estimate the location of this boundary. To explore the effect of resonances, we consider orbital element space near the conditions in the HD 82943 and 55 Cnc systems. Previous studies have shown that, for nonresonant systems, the two stability boundaries are nearly coincident. However, the Hill stability formulae are not applicable to resonant systems, and our investigation shows how the two boundaries diverge in the presence of a mean-motion resonance, while confirming that the Hill and Lagrange boundaries are similar otherwise. In resonance the region of stability is larger than the domain defined by the analytic formula for Hill stability. We find that nearly all known resonant interactions currently lie in this unexpectedly stable region, i.e., where the orbits would be unstable according to the nonresonant Hill stability formula. This result bears on the dynamical packing of planetary systems, showing how quantifying planetary systems' dynamical interactions (such as proximity to the Hill stability boundary) provides new constraints on planet formation models.

Subject headings: methods: *n*-body simulations — planetary systems

1. INTRODUCTION

By the end of 2006, 20 multiple planetary systems had been detected beyond the solar system (Butler et al. 2006; Wright et al. 2007). Of these, seven are likely to contain at least one pair that is in a mean-motion resonance. Barnes & Quinn (2004, hereafter BQ04) showed that one of these resonant pairs, HD 82943 b and c, had best-fit orbital elements that placed the system near a stability limit. Indeed, the best fit was unstable, but a small change (well within observational uncertainties) in the eccentricity e of the outer planet would make the system stable (BQ04; Ferraz-Mello et al. 2005). BQ04 also showed that stability requires the ratio of the orbital periods, P_c/P_b, to be near 2 and that the relative mean longitudes and difference in longitudes of pericenter lie in a range such that conjunctions never occur at the minimum distance between the orbits. This result suggests that, given the values of e and a of the two planets, stability is only possible if the two planets are in a 2 : 1 resonance.

Two types of stability have been considered in the literature. Hill stability requires the ordering of planets to remain constant for all time; the outer planet may escape to infinity. The equations that define the limits of Hill stability (i.e., Marchal & Bozis 1982; Gladman 1993) only apply to systems of two planets that are not in a resonance. Lagrange stability requires all planets to remain bound to the star, and the orbits evolve at least quasi-periodically. Lagrange stability is more meaningful, but its criteria have not been delineated analytically.

Barnes & Greenberg (2006a, hereafter BG06) showed that the Hill stability boundary is nearly the same as the Lagrange stability boundary, at least for the nonresonant planets in HD 12661 and 47 UMa. Although the Hill stability boundary was derived for nonresonant systems, it is not clear how mean-motion resonances distort it. Here we explore the stability boundary near two resonant systems, HD 82943 (Mayor et al. 2004) and 55 Cnc (Marcy et al. 2002; McArthur et al. 2004). Note that for both systems the inner planet of the resonant pair

is named b and the outer c. We find that the resonances do provide extra regions of Lagrange stability in phase space that extend beyond the analytic criterion. In § 2 we describe Hill and Lagrange stability and our numerical methods. In § 3 we present our results for HD 82943 and 55 Cnc. We also tabulate proximities to the Hill boundary for all applicable systems and find that all but one resonantly interacting pair would lie in an unstable region if not for the resonance. In § 4 we draw conclusions and suggest directions for future work.

2. METHODOLOGY

2.1. *Stability Boundaries*

Hill stability in a coplanar system can be described by the following inequality:

$$-\frac{2M}{G^2 M_*^3} c^2 h > 1 + 3^{4/3} \frac{m_1 m_2}{m_3^{2/3}(m_1 + m_2)^{4/3}}$$
$$- \frac{m_1 m_2 (11 m_1 + 7 m_2)}{3 m_3 (m_1 + m_2)^2} + ... , \quad (1)$$

where M is the total mass of the system, m_1 is the mass of the more massive planet, m_2 is the mass of the less massive planet, m_3 is the mass of the star, G is the gravitational constant, $M_* = m_1 m_2 + m_1 m_3 + m_2 m_3$, c is the total angular momentum of the system, and h is the energy (Marchal & Bozis 1982). If a given three-body system satisfies the inequality in equation (1), then the system is Hill stable. If this inequality is not satisfied, then the system may or may not be Hill stable. In this inequality, the left-hand side is a function of the orbits, but the right-hand side is only a function of the masses. This approach is fundamentally different from other common techniques for determining stability which exploit resonance overlaps (Wisdom 1982; Quillen & Faber 2006), chaotic diffusion (Laskar 1990; Pepe et al. 2007), fast Lyapunov indicators (Froeschlé et al. 1997; Sándor et al. 2007), or periodic orbits (Voyatzis & Hadjidemetriou 2005, 2006; Hadjidemetriou 2006).

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System	m_3 ($M_\odot$)	Planet	m ($M_{\rm Jup}$)	P (days)	e	ϖ (deg)	$T_{\rm peri}$ (JD)
HD 82943	1.05 ± 0.05[a]	b	0.88	221.6 ± 2.7	0.54 ± 0.05	138 ± 13	2451630.9 ± 5.9
		c	1.63	444.6 ± 8.8	0.41 ± 0.08	96 ± 7	2451620.3 ± 12.0
55 Cnc	0.95 ± 0.1[b]	b	0.84 ± 0.07	14.653 ± 0.0006	0.02 ± 0.02	99 ± 35	2450001.479 ± 10^{-6}
		c	0.21 ± 0.04	44.276 ± 0.021	0.339 ± 0.21	61 ± 25	2450031.4 ± 2.5
		d	4.05 ± 0.9	5360 ± 400	0.16 ± 0.06	201 ± 22	2785 ± 250

[a] Santos et al. (2000).
[b] Marcy et al. (2002).

BG06 use β (the left-hand side of eq. [1]) and $\beta_{\rm crit}$ (the right-hand side) to describe the Hill stability boundary. The Hill stability boundary is the curve defined by $\beta/\beta_{\rm crit} = 1$. BG06 showed that the Lagrange stability boundary appears to be located where $\beta/\beta_{\rm crit}$ is slightly larger than 1 (1.02 for 47 UMa and 1.1 for HD 12661).

2.2. *Numerical Methods*

For each system, HD 82943 and 55 Cnc, 1000 initial configurations were generated based on the observational data for each system (Mayor et al. 2004; Marcy et al. 2002); that is, the initial conditions spanned the range of observational uncertainty. Note that more recent, improved elements are available (Butler et al. 2006), but for our purposes the older values serve equally well. Orbital parameters that have known errors, such as e and the period P, are varied as a Gaussian centered on the best-fit value, with a standard deviation equal to the published uncertainty, and orbital elements are sampled appropriately. For elements with systematic errors, such as inclination, the initial conditions were varied uniformly. The inclination was varied between $0°$ and $5°$, and the longitude of ascending node was varied between 0 and 2π. Masses were then set to the observed mass divided by the sine of the inclination. The variation of orbits out of the fundamental plane will not significantly affect our calculations of Hill stability (Veras & Armitage 2004). Each element was varied independently. The distribution of initial conditions is presented in Table 1. In this table, ϖ is the longitude of periastron and $T_{\rm peri}$ is the time of periastron passage. The integrations were performed with SWIFT (Levison & Duncan 1994) and MERCURY (Chambers 1999), and conserve energy to at least 1 part in 10^4. For more details on these methods, consult BQ04.

For each simulation we numerically determined Lagrange stability on $\sim 10^6$ yr timescales. BQ04 showed that this timescale identifies nearly all unstable configurations. We then calculated $\beta/\beta_{\rm crit}$ in the parameter space sampled by the numerical integrations. Comparison of these two sets of results shows how the Hill and Lagrange stability boundaries are related near a mean-motion resonance.

3. RESULTS

For HD 82943 the "stability map" is shown by the gray-scale shading in Figure 1. Shading indicates the fraction of initial conditions, in a certain range of orbital element space, that give Lagrange stable behavior (no ejections or exchanges) over 10^6 yr: white bins contain only stable configurations, black only unstable, and the darkening shades of gray correspond to decreasing fractions of stable configurations. This representation plots the stable fraction as a function of two parameters: the eccentricity of the outer planer, e_c, and the ratio of the periods, P_c/P_b. The numerical simulations show that Lagrange stability is most likely for values of P_c/P_b slightly greater than 2 and e_c less than 0.4. BQ04 called this feature the "stability peninsula."

Superimposed on this gray-scale map are contours of $\beta/\beta_{\rm crit}$ values. All the values of $\beta/\beta_{\rm crit}$ are much less than 1.02, which ordinarily would imply instability. However, in the resonance zone where $P_c/P_b \approx 2$, the stability peninsula sticks into a regime ($\beta/\beta_{\rm crit}$ as small as 0.75) that would be unstable if the planets were not in a mean-motion resonance. Note that the numerical simulations include cases with variations of a few percent in mass; the $\beta/\beta_{\rm crit}$ contours shown are for the average mass but would shift only slightly over the range of masses.

For 55 Cnc, the stability map (Fig. 2) was developed from integrations over 4 million years, i.e., 10^6 orbits of the outer
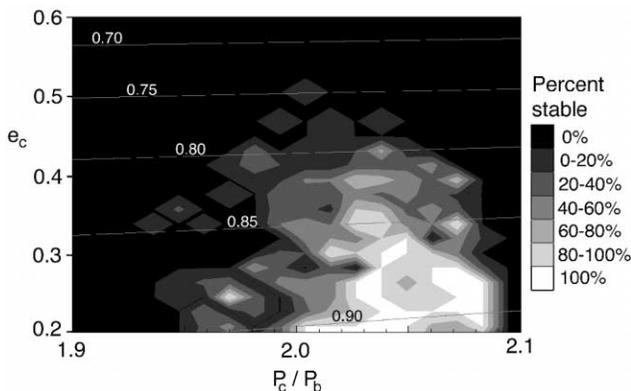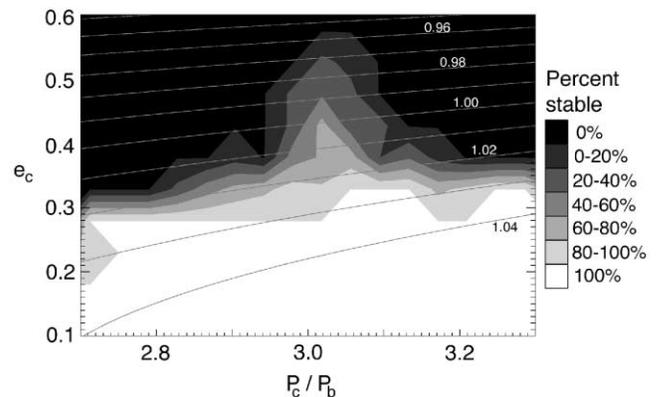

Fig. 1.—Stability map for HD 82943. The shading represents the fraction of simulations that are Lagrange stable on 10^6 yr timescales. The bin sizes are 0.02 for e_c and 0.01 for P_c/P_b. Contour lines show values of $\beta/\beta_{\rm crit}$. Ordinarily $\beta/\beta_{\rm crit} < 1.02$ would imply instability; however, the mean-motion resonance provides a stable region that would not exist if the resonance did not affect the motion.

Fig. 2.—Stability map for 55 Cnc (cf. Fig. 1). The bin sizes are 0.05 for e_c and 0.04 for P_c/P_b. As in HD 82943, the mean-motion resonance provides a larger Lagrange-stable region.

TABLE 2
VALUES OF $\beta/\beta_{\mathrm{crit}}$ FOR KNOWN SYSTEMS

System	Pair	$\beta/\beta_{\mathrm{crit}}$	Class
HD 202206[a]	b-c	0.883	R (5 : 1)
HD 128311[b]	b-c	0.968	R (2 : 1)
HD 82943[b]	b-c	0.946	R (2 : 1)
HD 73526[b]	b-c	0.982	R (2 : 1)
GJ 876[b]	c-d	0.99[c]	R (2 : 1)
47 UMa[b]	b-c	1.025	S
HD 155358[d]	b-c	1.043	S
HD 108874[b]	b-c	1.107	R (4 : 1)
υ And[b]	c-d	1.125[c]	S
HD 12661[b]	b-c	1.199	S
HIP 14810[e]	b-c	1.202	T
HD 169830[b]	b-c	1.280	S
HD 74156[b]	b-c	1.542	S
HD 190360[b]	b-c	1.701	T
HD 168443[b]	b-c	1.939	S
HD 38529[b]	b-c	2.070	S
HD 217107[b]	b-c	7.191	T

[a] Correia et al. (2005).
[b] Butler et al. (2006).
[c] An additional planet is present in this system.
[d] Cochran et al. (2007).
[e] Wright et al. (2007).

planet. Our simulations include planets b, c, and d, but not the inner planet e. Planet e is relatively small, and Zhou et al. (2004) found that the outer, nonresonant planet d does not appear to affect the global stability of the system. Therefore, our simulations should elucidate the relationship between Hill and Lagrange stability boundaries in the presence of a 3 : 1 mean-motion resonance.

Of our simulations, 50.2% $\pm$ 5.5% were Lagrange stable. The least massive planet, c, was the planet most likely to be ejected. In this system we see that stability is likely for $e_c < 0.3$ everywhere, except at $P_c/P_b \approx 3$, where it extends to $e \sim 0.55$.

Comparing this distribution with the analytical stability criterion ($\beta/\beta_{\mathrm{crit}} \lesssim 1.03$), we see that the numerical experiments reproduce that boundary, except in resonance where $\beta/\beta_{\mathrm{crit}}$ can be as low as 0.96. This stability peninsula for 55 Cnc does not protrude as far into the Hill unstable region as HD 82943. This difference may be because the 2 : 1 resonance is of a lower order (and thus stronger) than the 3 : 1 resonance, and therefore has a more pronounced stabilizing effect.

Next we tabulate $\beta/\beta_{\mathrm{crit}}$ values for all observed systems that contain two planets. We also include GJ 876 c and b, and υ And c and d. Equation (1) is only applicable to two-planet systems, but we consider these latter two pairs, which are each part of a bigger system, as the third planet in each system is probably too small or too far away to significantly change the interaction of those pairs. However, interpreting values of $\beta/\beta_{\mathrm{crit}}$ in systems of more than two companions should be made with caution, as there is no guarantee $\beta/\beta_{\mathrm{crit}} = 1$ corresponds to the Hill boundary for any individual pair.

Table 2 lists values of $\beta/\beta_{\mathrm{crit}}$ and the "class" of the interaction, which distinguishes the dominant phenomenon that changes the shapes of the orbits. "R" denotes pairs whose interaction is dominated by mean-motion resonances (Table 2 also lists the resonance), "T" indicates pairs that may have experienced significant tidal evolution, and "S" indicates pairs with strong secular interactions (Barnes & Greenberg 2006b). All but one resonantly interacting pair have $\beta/\beta_{\mathrm{crit}}$ values less than 1. If not for the resonance, these systems would be unstable.

Overall, we find that 70% of the pairs we consider are observed to have $\beta/\beta_{\mathrm{crit}} < 1.3$. HD 217107 is observed to have
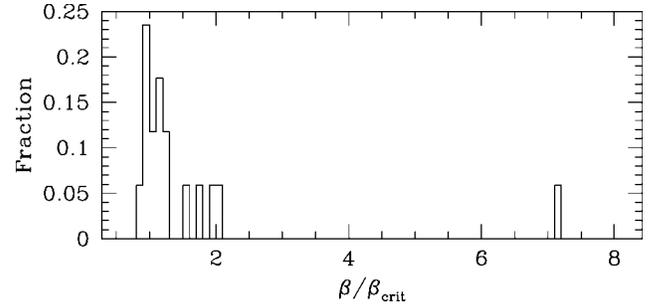

FIG. 3.—Histogram of values of $\beta/\beta_{\mathrm{crit}}$ (with a bin size of 0.1) for the 17 pairs of planets in Table 2.

$\beta/\beta_{\mathrm{crit}}$ significantly larger than other pairs. In Figure 3 we plot the distribution of $\beta/\beta_{\mathrm{crit}}$ values from Table 2.

4. CONCLUSIONS

By explicitly mapping how mean-motion resonances can provide additional regions of stability in orbital element space, we have found that nearly all observed resonant systems lie in these extended regions. More generally, we have also shown that the distribution of $\beta/\beta_{\mathrm{crit}}$ appears to show that many planetary systems (resonant or not) lie close to the limits of dynamical stability. These distributions provide new constraints for models of planet formation.

In the cases presented here, the 2 : 1 resonance provides a larger stable region than the 3 : 1 resonance, presumably because it is a lower order (stronger) resonance. However, for the 5 : 1 mean-motion resonance in HD 202206, $\beta/\beta_{\mathrm{crit}}$ can reach 0.88 and still be stable. So why is the range of stability for the 3 : 1 resonance in 55 Cnc so small? Perhaps if e_c in the 55 Cnc system has values in excess of 0.5, it does interact with the third planet, destabilizing the system. Future work should investigate the minimum $\beta/\beta_{\mathrm{crit}}$ that allows stability for each resonance. Future work may also determine if the peninsula we find in 55 Cnc is truncated due to interactions with 55 Cnc d.

We seek to identify the origin of the shape of the stability peninsulas in resonant systems. Ideally, a general expression will eventually be developed that describes the Lagrange stability boundary and that applies to planets both in and out of resonance. One avenue of research is to focus on close-approach distances. In the limit of zero eccentricity, orbits are unstable if they are separated by less than 3.5 mutual Hill radii (Gladman 1993). Therefore, we speculate that systems with approaches within this distance are unstable. For secularly evolving systems, the orbits change with time, and eventually the planets will line up at the minimum distance between the orbits over a secular period. Resonances can prevent planets from lining up in this danger zone, hence the stability peninsulas. This likely explanation for the shape of the Lagrange boundaries might be a fruitful direction of future research into planetary system stability.

The distribution of $\beta/\beta_{\mathrm{crit}}$ shows that, regardless of the presence of mean-motion resonance, many systems have values that are close to the stability boundary. This trend appears to support the hypothesis that planetary systems are dynamically "packed," i.e., that additional planets could not exist in orbits between those that are known without destabilizing the system (Barnes & Quinn 2001; BQ04; Barnes & Raymond 2004; Raymond & Barnes 2005; Raymond et al. 2006). Perhaps there is a minimum value of $\beta/\beta_{\mathrm{crit}}$ that would permit the insertion of an additional planet that leaves the system still stable. In other words it will be interesting to determine, for a given value of

$\beta/\beta_{\rm crit}$, the largest mass object that could orbit between two planets and still leave the system stable. Such a relation could produce an analytic criterion for dynamical packing, which can currently only be estimated numerically (e.g., Menou & Tabachnik 2003; Dvorak et al. 2003; Rivera & Haghighipour 2007).

Past work provides illumination on the possibility that some minimum value of $\beta/\beta_{\rm crit}$ may define the limit for which additional planets could be placed between the observed planets. The HD 168443 system ($\beta/\beta_{\rm crit} = 1.94$) has been shown to be unable to support even infinitesimal test masses (Barnes & Raymond 2004). The region between the known planets of HD 169830 ($\beta/\beta_{\rm crit} = 1.28$) is chaotic, and a planet in that region would most likely be unstable (Érdi et al. 2004). On the other hand, HD 38529 ($\beta/\beta_{\rm crit} = 2.07$) could support a Saturn-mass companion between the known planets (Barnes & Raymond 2004). These results suggest $\beta/\beta_{\rm crit} = 2$ may be the critical value.

The packing of the two planets in HD 190360 demands closer inspection. Although the orbits are more separated and less eccentric than those in HD 168443, their $\beta/\beta_{\rm crit}$ value (1.70) is less than that for HD 168443 (1.94). To explore this issue, we have integrated the HD 190360 system with a hypothetical Earth-mass planet on a circular orbit located at the midpoint between the apoastron distance of the inner planet and the periastron distance of the outer. The additional companion in the HD 190360 system survived for 10^6 yr. A similar experiment with HD 74156 ($\beta/\beta_{\rm crit} = 1.54$) showed ejection of the Earth-mass planet in only 2500 yr. We tentatively conclude that systems are packed if $\beta/\beta_{\rm crit} \lesssim 1.5$ and not packed if $\beta/\beta_{\rm crit} \gtrsim 2$, and the packing status is unknown in the range $1.5 \lesssim \beta/\beta_{\rm crit} \lesssim 2$. Future research needs to determine the relationship between $\beta/\beta_{\rm crit}$ and the possibility that additional planets could be stable between known ones.

As noted at the end of § 3, 70% of the tabulated systems have $\beta/\beta_{\rm crit} < 1.3$, indicating that the planets are too fully packed to allow any intermediate planets. This result, coupled with the limitations of radial velocity surveys to detect planets (e.g., we used minimum masses), suggests that the vast majority of multiple-planet systems are similarly fully packed. Our results are therefore consistent with the "packed planetary systems" hypothesis (BQ04; Barnes & Raymond 2004; Raymond & Barnes 2005; Raymond et al. 2006; see also Laskar 1996), which proposes that planetary systems tend to form so as to be dynamically packed. This hypothesis therefore predicts that HD 190360 and especially HD 217107 harbor additional undetected planets.

This investigation has identified a simple way to parameterize multiple-planet systems. At least for a two-planet system, a single parameter $\beta/\beta_{\rm crit}$ may indicate both stability and packing. Moreover, the statistics of the distribution of this dynamical parameter for observed systems are intriguing: planetary systems tend to be dynamically fully packed, and resonant systems lie at values of $\beta/\beta_{\rm crit}$ that would indicate instability for nonresonant systems. Describing planetary systems by parameterizing the character of their dynamical interaction is also the approach taken by Barnes & Greenberg (2006b), who calculated the proximities of planetary systems to the apsidal separatrix. These new methodologies focus on the proximities of the dynamical interactions to boundaries between qualitatively different dynamical regimes.

It now appears that about half of stars with planets have multiple planets (Wright et al. 2007), and descriptions of their dynamical interactions will therefore become increasingly more relevant, especially since many planets' eccentricities oscillate by 2 orders of magnitude (Barnes & Greenberg 2006b). We encourage research that models planet formation (e.g., Lee & Peale 2002; Sándor & Kley 2006) to include comparisons of the simulated values of $\beta/\beta_{\rm crit}$ to those of real planetary systems.

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