The Astrophysical Journal, 647: L163–L166, 2006 August 20


# STABILITY LIMITS IN EXTRASOLAR PLANETARY SYSTEMS

Rory Barnes[1] and Richard Greenberg[1]
*Received 2006 April 13; accepted 2006 July 10; published 2006 August 7*

## ABSTRACT

Two types of stability boundaries exist for any planetary system consisting of one star and two planets. Lagrange stability requires that the planets remain bound to the star, conserves the ordering of the distance from the star, and limits the variations of orbital elements like semimajor axis and eccentricity. Hill stability only requires that the ordering of the planets remain constant; the outer planet may escape to infinity. A simple formula defines a region in orbital element space that is guaranteed to be Hill-stable, although Hill-stable orbits may lie outside the region as well. No analytic criteria describe Lagrange stability. We compare the results of 1000 numerical simulations of planetary systems similar to 47 UMa and HD 12661 with these two types of boundaries. All cases are consistent with the analytic criterion for Hill stability. Moreover, the numerically determined Lagrange boundary lies close to the analytic boundary for Hill stability. This result suggests an analytic formulation that may describe the criterion for Lagrange stability.

*Subject headings:* methods: analytical — methods: *n*-body simulations — planetary systems — stars: individual (HD 12661, 47 Ursae Majoris)

## 1. INTRODUCTION

Since the discovery of the first extrasolar planetary system with multiple companions, $\upsilon$ And (Butler et al. 1999), substantial research has investigated the dynamics of multiplanet systems. Most of this work has examined the nature of individual systems such as 47 UMa (Fischer et al. 2002; Laughlin et al. 2002; Goździewski 2002) and HD 12661 (Fischer et al. 2003; Goździewski 2003; Lee & Peale 2003). One investigation, using numerical integration of orbits, showed that several of the known systems lie near an obvious stability boundary (Barnes & Quinn 2004).

The dynamical stability of gravitational systems of multiple (>2) particle systems has been studied for centuries. The description of the motions in this type of system have no analytic solution. Analytic constraints on dynamical stability began to emerge in the 1970s and 1980s, when it was shown that the motions of a system of two planets and a star would be bounded in some situations (Zare 1977; Szebehely 1980; Marchal & Bozis 1982; Milani & Nobili 1983; Valsecchi et al. 1984). These constraints can be interpreted in terms of the limitations on angular momentum exchange between the planets (Milani & Nobili 1983). However, this type of argument is only valid for two planets not involved in any low-order mean motion resonances. There is no known analytic boundary for systems in a low-order mean motion resonance, or a system with more than two planets.

Two predominant definitions of stability have emerged. In Hill (or hierarchical) stability, the ordering of the planets, in terms of distance from the central star, is conserved. However, the outermost planet may escape to infinity, and the system would still be considered stable. A more useful definition, called Lagrange stability, is more stringent: the planets remain bound to the central star, changes in the ordering of the planets are forbidden, and the semimajor axis and eccentricity variations also remain bounded.

Currently, investigations of the Lagrange stability of a system are generally made through numerical simulations (e.g., Barnes & Quinn 2004). However, Marchal & Bozis (1982) noted that: "Some studies (Szebehely & McKenzie 1977; Szebehely 1978, 1980) seem to show a correlation between the Hill stability and the other types of stability related to escape and exchanges; it would be interesting to investigate these questions." In 1982 limited computer power made such an investigation daunting. Now with modern computing power and motivated by exoplanet systems we can revisit their supposition.

Gladman (1993) extended the study of Hill stability by approximating the boundary (see § 2) in orbital element space. He verified the analytic expression through numerical tests, in certain limits. More recently, Veras & Armitage (2004) modified the Hill criterion for application to mutually inclined orbits.

Other stability studies consider boundaries between periodic, quasi-periodic, and formally chaotic orbits via the fast Lyapunov indicator (Froeschlé et al. 1997). Such boundaries have been explored in extrasolar planetary systems (Goździewski et al. 2001, 2002, 2003; Kiseleva-Eggleton et al. 2002); however, it is not clear how these boundaries (or any other test based on Lyapunov exponents) relate to limits of Lagrange stability.

In this Letter, we compare the analytic description of *Hill* stability to a numerical determination of *Lagrange* stability. In § 2 we review the Hill stability equations for systems of two planets. In § 3 we numerically test the analytic solutions and compare the predictions of Hill stability with Lagrange stability, which is determined by *N*-body simulations. In § 4 we draw general conclusions and suggest directions for future work.

## 2. HILL STABILITY

There is no analytic solution for the motion of three gravitating bodies, but in certain situations the range of motion can be shown to be bounded; certain regions of phase space are forbidden for each particle (Marchal & Bozis 1982; Milani & Nobili 1983; Valsecchi et al. 1984). This boundary is a direct result of the conservation of angular momentum. For the case of two planets around a much more massive star, the eccentricity exchange (through exchange of orbital angular momentum) is limited, and the planets will never experience a close enough encounter to expel the interior planet from the system (i.e., Hill stability).

[1] Lunar and Planetary Laboratory, University of Arizona, Tucson, AZ 85721; rory@lpl.arizona.edu.

TABLE 1
Orbital Elements and Errors

| System | $m_3$ ($M_\odot$) | Planet | $m$ ($M_J$) | $P$ (days) | $e$ | $\varpi$ (deg) | $T_{peri}$ (JD − 2,450,000) |
|---|---|---|---|---|---|---|---|
| HD 12661 ...... | 1.07 | b | 2.3 | 263.3 (20) | 0.35 (0.1) | 292.6 (20) | 9943.7 (10) |
| | | c | 1.56 | 1444.5 (75) | 0.2 (0.1) | 147 (20) | 9673.9 (40) |
| 47 UMa ........ | 1.03 | b | 2.54 | 1089 (3) | 0.06 (0.014) | 172 (15) | 3622 (34) |
| | | c | 0.76 | 2594 (90) | 0.1 (0.1) | 127.0 (56.0) | 1363.5 (493) |

Marchal & Bozis (1982) quantified the criterion for Hill-stable configurations as

$$-\frac{2M}{G^2M_*^3}c^2h > 1 + 3^{4/3}\frac{m_1m_2}{m_3^{2/3}(m_1+m_2)^{4/3}} - \frac{m_1m_2(11m_1+7m_2)}{3m_3(m_1+m_2)^2} + \dots, \qquad (1)$$

where $M$ is the total mass of the system, $m_1$ and $m_2$ are the planet masses (the subscript 1 refers to the inner planet), $m_3$ is the mass of the star, $G$ is the gravitational constant, $M_* = m_1m_2 + m_1m_3 + m_2m_3$, $c$ is the total angular momentum of the system, and $h$ is the energy. If a given three-body system satisfies the inequality in equation (1), then the system is said to be Hill-stable, and close approaches are forbidden *for all time*. If the inequality fails to be satisfied, then the Hill stability of the system is unknown; *the system may still be Hill-stable*. Note that the left-hand side of equation (1) is a function of the positions and velocities of the system, and the right-hand side is purely a function of the masses. Thus, for given masses, equation (1) defines a boundary in orbital element space.

Gladman (1993) showed that equation (1) could be changed to barycentric orbital elements and rewritten, to first order, as

$$\alpha^{-3}\left(\mu_1 + \frac{\mu_2}{\delta^2}\right)(\mu_1\gamma_1 + \mu_2\gamma_2\delta)^2 > 1 + 3^{4/3}\frac{\mu_1\mu_2}{\alpha^{4/3}}, \qquad (2)$$

where $\mu_i = m_i/M$, $\alpha = \mu_1 + \mu_2$, $\gamma_i = (1 - e_i^2)^{1/2}$, $\delta = (a_2/a_1)^{1/2}$, $e$ is the eccentricity, $a$ is the semimajor axis, and $i = 1, 2$. For given masses and eccentricities, there is a critical value of the semimajor axis ratio (or equivalently a critical value of $\delta$, which we call $\delta_{crit}$), for which the two sides of equation (2) are equal. If $a_2/a_1$ is large enough (i.e., $\delta > \delta_{crit}$), then the system is surely Hill-stable—otherwise, maybe not.

The boundary for Lagrange stability should lie at $\delta > \delta_{crit}$ (larger orbital separation) because it is a more stringent definition of stability. As we show below, this expectation is borne out by our numerical integrations. There would be no reason to expect, a priori, that the actual Lagrange boundary would be correlated with the Hill boundary limit. There might not even be a clear boundary in orbital element space between Lagrange-stable and Lagrange-unstable configurations.

## 3. STABILITY OF EXOPLANET SYSTEMS

In this section we numerically explore the stability of hypothetical systems with masses and orbital elements similar to the 47 UMa and HD 12661 systems. In Table 1 we present the current best fits (masses and orbits) and errors, shown in parentheses, for each of these two systems. In this table $m$ is the planetary mass, $\varpi$ is the longitude of periastron, and $T_{peri}$ is the time of periastron passage. Equations (1)–(2) should apply to these systems because each has only two planets not in low-order mean motion resonance. Moreover, we can exploit orbital integrations that had already been performed for different purposes (Barnes & Quinn 2003, 2004). The numerical simulations were performed with MERCURY6 (Chambers 1999) for HD 12661 or SWIFT (Levison & Duncan 1994) for 47 UMa.

For each of the two systems, Barnes & Quinn (2003, 2004) considered 1000 different initial conditions distributed over the range of observational uncertainty (Fischer et al. 2002, 2003). For most orbital elements they selected values at random from a Gaussian distribution. However, the inclinations were selected from a uniform distribution between 0° and 5°, and the longitude of ascending node from a uniform distribution from 0 to $2\pi$. For the initial conditions each orbital element was selected independently. This distribution is not ideal for mapping stability; a priori, a uniform distribution, far from any mean motion resonances, might have been more efficient, except that we already had these results in hand.

### 3.1. *HD 12661*

For HD 12661 the outcome after 4 million years of each numerical experiment (Lagrange stability or instability) is shown as a function of $e_b$, $e_c$, and $a_c/a_b$ in Figure 1. This choice of timescale is somewhat arbitrary but has been shown to identify most unstable configurations (Ford et al. 2001; Barnes & Quinn 2004). Also shown in Figure 1, for comparison with the numerical results, is the surface represented by equation (2). According to equation (2), all configurations that lie to the lower left of the curves (smaller eccentricities) *must* be Hill-stable. Note that this criterion is not exclusive: Hill-stable configurations are possible outside that region as well. Therefore, the actual boundary between Hill stability and instability lies to the upper right of the curves. The Lagrange boundary must lie below and to the actual Hill boundary of the curves because Lagrange stability is a more stringent criterion.

In these results every case considered remained Hill-stable over 4 million years (Fig. 1, *crosses and circles*) consistent with the expectations of equation (2). Therefore, regardless of Lagrange stability, these configurations were all Hill-stable. In principle, any case that is Hill-stable and Lagrange-unstable (Fig. 1, *circles*) could have gone Lagrange-unstable either by switching the planets' order or by ejecting the outer planet. Every Lagrange-unstable configuration of HD 12661 ejected the outer planet (planet c). Most interestingly, the boundary of Lagrange stability is close to, and tracks, the surface defined by equation (2), which was derived in the context of Hill stability. Marchal & Bozis (1982) had suspected such a relationship.

Next let us quantify how far the numerically determined Lagrange boundary is from equation (2). For each configuration we determine the value of $\delta/\delta_{crit}$. We then plot as a function of $\delta/\delta_{crit}$ the fraction, $f$, in each bin that is Lagrange-stable over 4 million years (Fig. 2). There is a sudden transition (independent of eccentricity) from Lagrange-unstable configurations

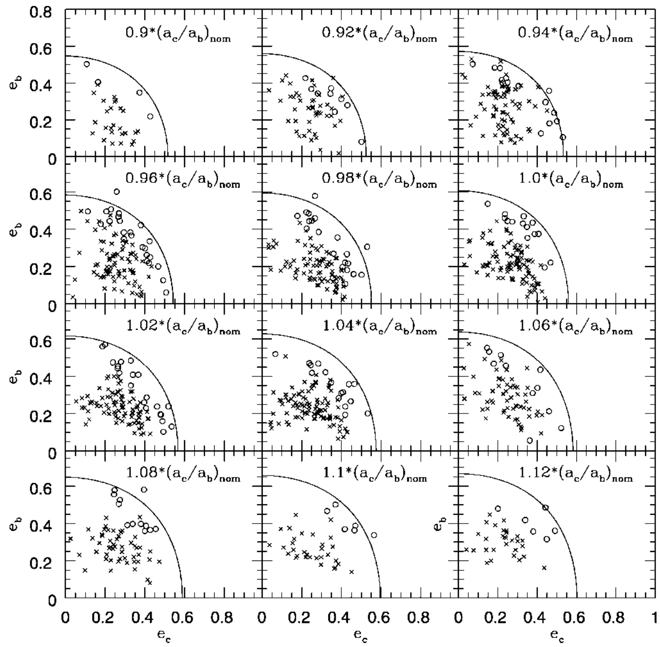


FIG. 1.—Lagrange stability of different initial configurations of the HD 12661 planetary system based on $e_b$, $e_c$, and $a_c/a_b$. Each panel is a slice through this three-dimensional space, showing all cases that begin with $a_c/a_b$ within 1% of the value listed at the top of the panel. [Here the ratio $(a_c/a_b)_{\rm nom}$ is the best-fit ratio of the semimajor axes, which is 3.08 for HD 12661. For example, the top left panel contains all trials that began with a ratio of $a_c/a_b$ in the range 0.89–0.91 times the best-fit value.] Crosses represent Lagrange-stable configurations, circles are Lagrange-unstable configurations, and the solid line represents the solution to eq. (2).

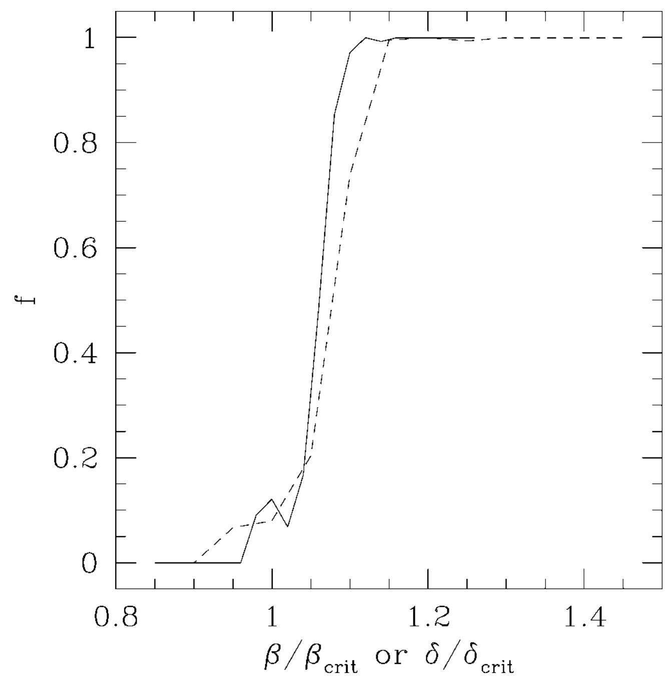


FIG. 2.—Fraction of cases that are Lagrange-stable as a function of proximity to the Hill stability boundary. In this plot we also compare of the exact solution to the Hill stability boundary, eq. (1) (*solid line*), to that of the approximate solution, eq. (2) (*dashed line*), for the simulations of HD 12661. The transition from instability to stability occurs at values slightly greater than 1.

to Lagrange-stable near $\delta/\delta_{\rm crit} = 1.05$. The Lagrange stability boundary lies close to the surface defined by equation (2).

Equation (1) can also be compared with the results of our numerical simulations. The left-hand side of equation (1) is a function of orbital elements that we call $\beta$. Figure 2 also includes a plot of $f$ as a function of $\beta/\beta_{\rm crit}$ ($\beta_{\rm crit}$ being the right-hand side of eq. [1]), showing a clear transition within about 5% of the boundary defined by equation (1). Even though the initial eccentricities may be large (some are over 0.5) the approximate solution, equation (2), appears to be in good agreement with equation (1). Both the $\beta$ and $\delta$ curves in Figure 2 show that there is a relatively narrow transition from Lagrange stability to Lagrange instability.

For the best-fit values to the observed HD 12661 system, we find that $\delta = 1.756$ and $\delta_{\rm crit} = 1.476$. The ratio is 1.19, putting this system within, but not deep within, the stable zone.

### 3.2. *47 UMa*

Figure 3 shows results for 47 UMa in a similar format as Figure 1. Results here are based on a $10^6$ yr timescale, which was shown to be a sufficient timescale to determine stability (Barnes & Quinn 2004). The eccentricity ranges differ from one another (and from those in Fig. 1) because the uncertainties in the two eccentricities are different (see Table 1). As in HD 12661, we see that the Lagrange stability limit lies just inside the curve for equation (2).

Figure 4 shows the fraction of Lagrange stable configurations (from Fig. 3) as a function of $\beta/\beta_{\rm crit}$ and $\delta/\delta_{\rm crit}$. As with HD 12661, the transition to Lagrange stability is at values of $\beta/\beta_{\rm crit}$ and $\delta/\delta_{\rm crit}$ only slightly greater than 1. Also, like HD 12661, every Lagrange-unstable configuration ejected the outer planet, confirming the Hill stability criterion. Once again we see that the Lagrange stability boundary appears to track the surface defined by equation (2). Unlike HD 12661 the two curves do not track each other exactly, but they are within a few percent, consistent with the accuracy of the approximation of equation (2).

For the best-fit values to the observed 47 UMa system, we find that $\delta = 1.336$ and $\delta_{\rm crit} = 1.195$. Therefore, the ratio of the two is 1.117, and the system is probably stable.

## 4. CONCLUSIONS

Although equation (1), and its equivalent equation (2), were derived in the context of Hill stability, we have found that it appears to be a good predictor of Lagrange stability, confirming the suspicions of Marchal & Bozis (1982). At this point we tentatively conclude that if $\delta \gtrsim 1.1\delta_{\rm crit} \equiv \delta_{\rm LS}$, then a two planet system is Lagrange-stable. In terms of $\beta/\beta_{\rm crit}$, Lagrange stability appears to be guaranteed at slightly smaller values. Additional work that numerically integrates various hypothetical systems is needed to test the validity of these empirical results. At this point, however, we tentatively find that if the ratio of the semimajor axes were 1% and 4% closer for 47 UMa and HD 12661, respectively, then the Lagrange stability of the systems could not be guaranteed. For these two systems, then, we have quantified how far each is from the Lagrange stability boundary.

Equations (1) and (2) were derived in the context of Hill stability, but they provide only weak constraints. They do not actually define the boundary between Hill stability and instability. Ironically, it now appears that these equations actually approximate the boundary of Lagrange stability. We would encourage a search for the explanation for this somewhat surprising correlation between equation (2) and the actual Lagrange limit. If a physical explanation could be identified, then

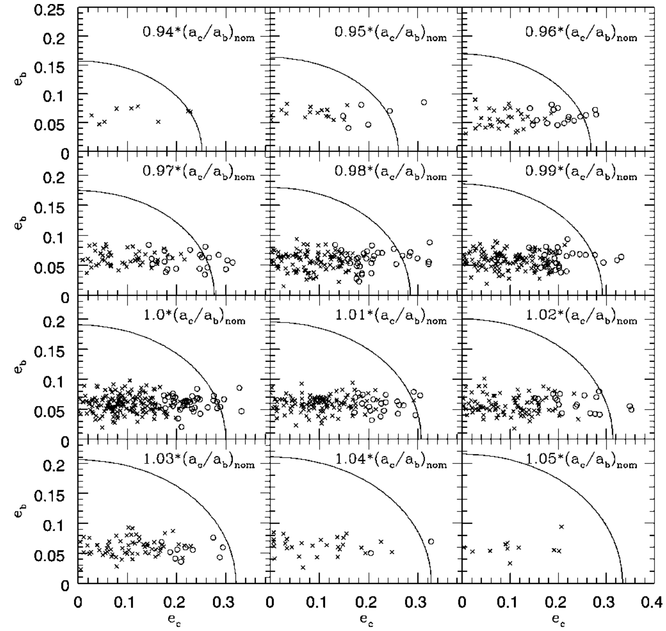


FIG. 3.—Lagrange stability of different initial configurations of the 47 UMa planetary system based on $e_b$, $e_c$, and $a_c/a_b$. The format is the same as Fig. 1 except each panel is a slice through the parameter space showing cases that began within 0.5% of the value of $a_c/a_b$ listed at the top of the panel. For 47 UMa the nominal ratio of the semimajor axes is 1.78. Crosses represent Lagrange-stable configurations, circles are Lagrange-unstable configurations, and the solid line represents the solution to eq. (2).

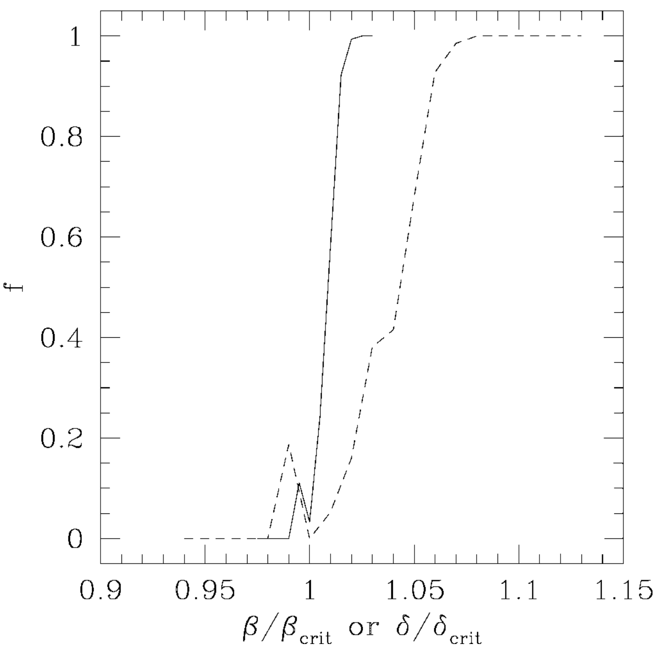


FIG. 4.—Lagrange stability of configurations of 47 UMa as a function of $\beta/\beta_{\rm crit}$ (*solid line*) and $\delta/\delta_{\rm crit}$ (*dashed line*). As with the HD 12661 system, Lagrange stability occurs at $\delta \approx 1.1\delta_{\rm crit}$.

it may allow a quantification of Lagrange stability for an arbitrary number of planets. In addition, it would be interesting to see if the inclined Hill equation (Veras & Armitage 2004) also tracks Lagrange stability. Future numerical work may also determine how close equation (2) is to the actual Hill limit.

The nature of Lagrange stability is a pressing issue given the proximity of several systems to the boundary between Lagrange stability and instability (Barnes & Quinn 2001, 2004; Goździewski 2002, 2003). The proximities of these systems to Lagrange instability have led to the "packed planetary systems" hypothesis (Barnes & Quinn 2004; Barnes & Raymond 2004; Raymond & Barnes 2005, Raymond et al. 2006; see also Laskar 2000), which suggests that all planetary pairs formed close to the Lagrange stability limit. The verification or rejection of this hypothesis hinges on both theoretical advances (such as this quantitative description of the Lagrange boundary) and observational improvements (such as breaking the so-called mass-inclination degeneracy and reductions in orbital element errors). The results presented here might represent the first step toward a theoretical understanding of the packing of planetary systems.

This work was funded by NASA's Planetary Geology and Geophysics program grant number NNG05GH65G. We would also like to thank an anonymous referee for suggestions that greatly clarified this manuscript.

## REFERENCES

Barnes, R., & Quinn, T. 2001, ApJ, 550, 884
———. 2003, BAAS, 35, 1041
———. 2004, ApJ, 611, 494
Barnes, R., & Raymond, S. N. 2004, ApJ, 617, 569
Butler, R. P., et al. 1999, ApJ, 526, 916
Chambers, J. E. 1999, MNRAS, 304, 793
Fischer, D., et al. 2003, ApJ, 586, 1394
———. 2002, ApJ, 564, 1028
Ford, E. B., Havlickova, M., & Rasio, F. 2001, Icarus, 150, 303
Froeschlé, C., Lega, E., & Gonzci, R. 1997, Celest. Mech. Dyn. Astron., 67, 41
Gladman, B. 1993, Icarus, 106, 247
Goździewski, K. 2002, A&A, 393, 997
———. 2003, A&A, 398, 1151
Goździewski, K., et al. 2001, A&A, 378, 569
Kiseleva-Eggleton, L., et al. 2002, ApJ, 578, L145
Laskar, J. 2000, Phys. Rev. Lett., 84, 3240
Laughlin, G., Chambers, J., & Fischer, D. 2002, ApJ, 579, 455
Lee, M. H., & Peale, S. J. 2003, ApJ, 592, 1201
Levison, H. F., & Duncan, M. J. 1994, Icarus, 108, 18
Marchal, C., & Bozis, G. 1982, Celest. Mech., 26, 311
Milani, A., & Nobili, A. M. 1983, Celest. Mech., 31, 213
Raymond, S. N., & Barnes, R. 2005, ApJ, 619, 549
Raymond, S. N., Barnes, R., & Kaib, N. 2006, ApJ, 644, 1223
Szebehely, V. 1978, Celest. Mech., 18, 383
———. 1980, Celest. Mech., 22, 7
Szebehely, V., & Mackenzie, R. 1977, AJ, 82, 79
Valsecchi, G. B., Carusi, B. A., & Roy, A. 1984, Celest. Mech., 32, 217
Veras, D., & Armitage, P. 2004, Icarus, 172, 349
Zare, K. 1977, Celest. Mech., 16, 35