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EXTRASOLAR PLANETARY SYSTEMS NEAR A SECULAR SEPARATRIX

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ABSTRACT

Extrasolar planetary systems display a range of behavior that can be understood in terms of the secular theory of classical celestial mechanics, including the motions of the major axes. Four planet pairs in the seventeen known extrasolar planetary systems with multiple planets (υ And, 47 UMa, 55 Cnc, and HD 128311), have trajectories in orbital element space that lie close to the separatrix between libration and circulation. Here we examine the dynamics of the first two, which are not in mean motion resonance. The basics of secular theory are reviewed in order to develop insight into this behavior. The definition of a secular resonance is discussed, correcting misconceptions in the literature; it is not synonymous with libration and is not a commensurability of eigenfrequencies. The behavior of these two near-separatrix systems is evaluated with updated orbital elements by comparing both analytical and numerical results. We find that the apsidal motion from secular theory does not match the predictions from N-body simulations and conclude that first-order secular theory should be used with caution on extrasolar planetary systems. While the existence of one near-separatrix system could be explained simply by chance initial conditions, the fact that there are several is improbable unless some physical process tends to set up systems near the separatrix. Explanations based on an impulsive increase in the eccentricity of one planet are promising, but key issues remain open.

Subject headings: celestial mechanics — planetary systems

1. INTRODUCTION

The wide range of morphologies of extrasolar planetary systems has provided new examples of librational behavior, including types of behavior well known from classical celestial mechanics, that may contribute to their long-term stability. Determination of the stability of a particular system and/or whether it is locked into a librating state can be complicated by two factors: observational uncertainties in the orbital parameters, and lack of precision in orbital theory. Often the range of observational uncertainty includes regions of parameter space for which a given system may be unstable, perhaps indicating a tendency for planets to form so densely packed that they lie near the limits of stability (Barnes & Quinn 2004). Similarly, we show in this paper that there is some indication from the limited (but growing) number of known systems that there is a tendency for them to be near boundaries between modes of mutual apsidal motion. This boundary is often called the "separatrix." Here we use the term separatrix to refer to the boundary between trajectories in phase space of libration and circulation. A more restricted definition would reserve the term for cases in which the trajectories on opposite sides diverge, which is not the case in the systems discussed here, as we show below. Two examples for which the behavior of the relative orientations of major axes are close to the separatrix between libration and circulation through 360° are υ Andromedae (Butler et al. 1999; Ford et al. 2005) and 47 Ursae Majoris (Fischer et al. 2002; Laughlin et al. 2002, hereafter LCF02). In addition, two mean motion resonance systems (55 Cnc b and c, and HD 128311) also show apsidal motion near the separatrix (Greenberg & Barnes 2005).

The tendency for systems to lie near the separatrix is not fully understood. For υ And, Ford et al. (2005) proposed that one planet suddenly acquired a substantial orbital eccentricity, perhaps due to an encounter with another, now-escaped planet. The interactions between the newly eccentric planet and an interior planet led to periodic oscillations of both eccentricities. If the inner planet started with a circular orbit, secular theory would require that the behavior track the separatrix between libration and circulation, such that its eccentricity would periodically return to zero.

According to LCF02, the 47 UMa system does not lie near the separatrix; it librates in an aligned configuration. This determination was based on a criterion from classical celestial mechanics. The authors used numerical integrations to confirm the libration. In this paper, however, we show that their numerical confirmation of the apsidal motion of 47 UMa was incorrect. Now, further observations of 47 UMa by the California and Carnegie Planet Search[2] (PI: G. Marcy & P. Butler; as of 2005 February 6) have refined the orbital elements, and the new values are significantly different from those assumed by LCF02. It should also be noted that Naef et al. (2004) were unable to corroborate the existence of planet c. However, here we presume that the second planet does exist, and will reconsider the orbital behavior of 47 UMa with these updated orbital elements. We label the elements from Fischer et al. (2002) as "old" and those from Planet Search as "new."

The changing best estimates of the elements suggests that it is useful to have a general map of the boundaries in orbital element space between libration and circulation, and then consider real systems in this context. Here, we first review (in § 2) the analytical basis of secular theory to show how the boundaries are defined in classical celestial mechanics. Most sources cited in the literature (e.g., Brouwer & Clemence 1961; Rasio 1995; Murray & Dermott 1999; Zhou & Sun 2003) use a matrix-based solution to determine the eigenvectors and eigenfrequencies, with the required matrix inversion carried out numerically. However, that approach obscures the dependence of the results

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(including various modes of behavior) on key parameters of the system. In our exposition, we obtain the eigenfrequencies and eigenvectors algebraically and discuss their implications. For example, we show the distinction between the concepts of secular resonance and libration of major-axis orientations, which can be associated but need not be.

Then, in § 3, we apply numerical integrations to the same specific orbital elements for 47 UMa assumed by LCF02 to show the significant differences between analytical and numerical results, which are due to various approximations in the analytical methods. While the analytical results do show libration, as shown in LCF02, the numerical ones indicate circulation, contrary to their interpretation. We also use numerical integrations to refine the map in orbital element space of conditions for libration versus circulation, using a format introduced in LCF02. Surprisingly, when we consider the most recently determined best-fit elements for the 47 UMa system, we find that it still lies near the separatrix, but now the analytical criterion indicates circulation, while the numerical integration shows libration.

Finally in § 4, we consider possible origins for near-separatrix behavior. We conclude, from a simple model, that these systems' proximities to the separatrix are not likely by chance and do require a physical explanation. The planet-planet scattering model of Malhotra (2002) and Ford et al. (2005) is promising but key issues remain unresolved.

2. SECULAR THEORY

2.1. *General Solution*

In this section we review the basics of secular theory. We define the following variables: $h = e\sin(\varpi)$ and $k = e\cos(\varpi)$, where e is the eccentricity and ϖ is the longitude of periastron. To distinguish between the planets we will label the outer planet with a prime. The disturbing function for the effects of m' on m, where m represents mass, is

$$R = \frac{Gm'}{4a'}\left[\frac{\alpha}{2}b_{3/2}^{(1)}(\alpha)e^2 - \alpha b_{3/2}^{(2)}(\alpha)ee'\cos(\varpi' - \varpi)\right], \quad (1)$$

where G is the gravitational constant, a is semimajor axis, α is the ratio of semimajor axes, and b is the Laplace coefficient as defined in Murray & Dermott (1999). In equation (1), terms of higher order in eccentricities have been neglected, so the theory will be less accurate for systems with large values of e or e'. Similarly, the disturbing function for m on m' is

$$R' = \frac{Gm}{4a'}\left[\frac{\alpha}{2}b_{3/2}^{(1)}(\alpha)e'^2 - \alpha b_{3/2}^{(2)}(\alpha)ee'\cos(\varpi' - \varpi)\right]. \quad (2)$$

Noting that $e^2 = h^2 + k^2$ and $ee'\cos(\varpi' - \varpi) = hh' + kk'$, we can rewrite equations (1) and (2) as

$$R = \frac{Gm'}{4a'}\left[\frac{\alpha}{2}b_{3/2}^{(1)}(\alpha)(h^2 + k^2) - \alpha b_{3/2}^{(2)}(\alpha)(hh' + kk')\right], \quad (3)$$

$$R' = \frac{Gm}{4a'}\left[\frac{\alpha}{2}b_{3/2}^{(1)}(\alpha)(h'^2 + k'^2) - \alpha b_{3/2}^{(2)}(\alpha)(hh' + kk')\right]. \quad (4)$$

Note that secular theory assumes that there are no mean motion resonances and that the short-period variations of the mean motions can be neglected. We also assume inclination effects are negligible, which is reasonable because the coupling of e and i occurs at higher order.

In general the equations for the variation in h and k are

$$\frac{dh}{dt} = +\frac{na}{GM}\frac{\delta R}{\delta k}$$
$$\frac{dk}{dt} = -\frac{na}{GM}\frac{\delta R}{\delta h}. \quad (5)$$

We can therefore rewrite these equations (now including the outer planet) as

$$\frac{dh}{dt} = Ak - Bk'$$
$$\frac{dk}{dt} = -Ah + Bh'$$
$$\frac{dh'}{dt} = -Ck + Dk'$$
$$\frac{dk'}{dt} = Ch - Dh', \quad (6)$$

where

$$A = \frac{nm'}{4M}\alpha^2 b_{3/2}^{(1)}(\alpha)$$
$$B = \frac{nm'}{4M}\alpha^2 b_{3/2}^{(2)}(\alpha)$$
$$C = \frac{n'm}{4M}\alpha b_{3/2}^{(2)}(\alpha)$$
$$D = \frac{n'm}{4M}\alpha b_{3/2}^{(1)}(\alpha), \quad (7)$$

where M is the central mass, and n is the mean motion. In order to keep this analysis as general as possible, we obtain the solution of the set of differential equations in terms of A, B, C, and D. From inspection of equation (6), the solution must be of the form

$$h = a\sin(\omega t + \delta)$$
$$k = a\cos(\omega t + \delta)$$
$$h' = c\sin(\omega t + \delta)$$
$$k' = c\cos(\omega t + \delta). \quad (8)$$

Inserting equation (8) into equation (6) yields only two independent equations:

$$a\omega = Aa - Bc,$$
$$c\omega = -Ca + Dc. \quad (9)$$

Solving each equation for ω yields

$$\omega = A - B/K = -CK + D, \quad (10)$$

where $K \equiv a/c$.

We now have two equations and two unknowns, ω and K. We cannot solve for the values of a and c individually, only their ratio K. Solving for K in equation (10) yields

$$K_{\pm} = \frac{D - A \pm \sqrt{(D - A)^2 + 4CB}}{2C}. \quad (11)$$

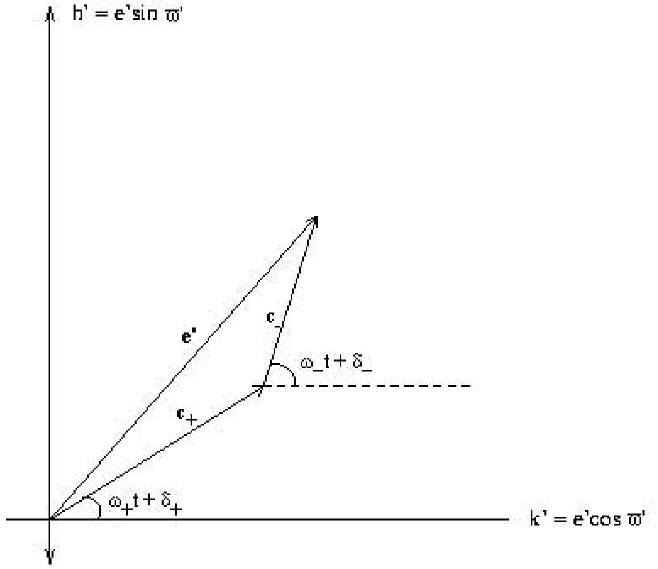

Fig. 1.—Schematic of $\boldsymbol{e}'$, the vector sum of two vectors, $\boldsymbol{c}_+$ and $\boldsymbol{c}_-$, which make angles $(\omega_+ t + \delta_+)$ and $(\omega_- t + \delta_-)$ with the k-axis, respectively.

Let K_+ and K_- be the two solutions corresponding to the + and − operation in front of the square root. Inserting equation (11) into equation (10) we obtain

$$\omega_\pm = -\frac{1}{2}\left[-A - D \pm \sqrt{(D-A)^2 + 4CB}\right]. \qquad (12)$$

The frequency with the positive square root (ω_+) corresponds to the K_+ solution, and ω_- corresponds to the K_- solution. Thus the general solution to equation (8) becomes

$$\begin{aligned} h &= K_+ c_+ \sin(\omega_+ t + \delta_+) + K_- c_- \sin(\omega_- t + \delta_-) \\ k &= K_+ c_+ \cos(\omega_+ t + \delta_+) + K_- c_- \cos(\omega_- t + \delta_-) \\ h' &= c_+ \sin(\omega_+ t + \delta_+) + c_- \sin(\omega_- t + \delta_-) \\ k' &= c_+ \cos(\omega_+ t + \delta_+) + c_- \cos(\omega_- t + \delta_-), \end{aligned} \qquad (13)$$

where the amplitudes, c_+ and c_-, and the corresponding phases, δ_+ and δ_-, are the four constants of integration of the four first-order differential equations in equation (6).

2.2. *Trajectories in Orbital Element Space*

In order to visualize this solution, consider a polar-coordinate plot of $\boldsymbol{e}'$ versus ϖ', which is equivalent to a Cartesian plot of h' versus k' (Fig. 1). The position (k', h') represents a vector (which we call $\boldsymbol{e}'$) whose magnitude is e' and direction is ϖ', the direction of periastron. According to equation (13), $\boldsymbol{e}'$ is the sum of two vectors, one of magnitude c_+ with direction $(\omega_+ t + \delta_+)$ and the other of magnitude c_- with direction $(\omega_- t + \delta_-)$. We call these two vectors $\boldsymbol{c}_+$ and $\boldsymbol{c}_-$ and note that each has a constant magnitude and a uniformly varying direction. Their sum $\boldsymbol{e}'$ continually varies in magnitude and direction.

If we were to plot the vector $\boldsymbol{e}$, defined as (k, h) on the same plot, it too would be the sum of two vectors; one with the same direction as $\boldsymbol{c}_+$ but with its magnitude changed by a factor of K_+, and the other with the same direction as $\boldsymbol{c}_-$ but with its magnitude changed by a factor of K_-.

2.3. *Numerical Example (47 UMa)*

As an example, consider the behavior of 47 UMa as modeled by LCF02, which was based on the orbital elements of Fischer et al. (2002; see our Table 1). In our Figure 2 we reproduce Figure 2 from LCF02, in which the evolution of $\Delta\varpi$, e, and e' is plotted from both a secular solution and a numerical simulation. LCF02's figure shows results from analytical secular theory and from numerical integrations. They reported that both methods give similar results, although we do see some differences; most significant is that $\Delta\varpi$ reaches to or near 180°. The dots near zero, while the system is close to $|\Delta\varpi| = 180°$, result from plotting the N-body parameters in astrocentric coordinates. As secular theory provides orbital elements in this coordinate system, the numerical results were also plotted in this system. We show in § 3 that this is an artifact of this choice of coordinate system and that the system is indeed circulating. In this section we compare results from our solution (§§ 2.1 and 2.2) with the results of LCF02.

With the numerical values from Table 1, we find that $A = 4 \times 10^{-4}$ yr^{-1}, $B = 2.7 \times 10^{-4}$ yr^{-1}, $C = 6.7 \times 10^{-4}$ yr^{-1}, and $D = 10^{-3}$ yr^{-1}. Therefore, $K_+ \approx 1.23$ and $K_- \approx -0.33$, with corresponding (k, h) vector rotation rates $\omega_+ = 1.85 \times 10^{-4}$ yr^{-1} and $\omega_- = 12.2 \times 10^{-4}$ yr^{-1}. LCF02 found that periodically the system reaches a condition in which $\Delta\varpi = 0$, e' is nearly zero, and e is about 0.06. Let us define this moment as $t = 0$, and let ϖ (the direction of $\boldsymbol{e}$ at $t = 0$) define the reference direction (the k-axis). We plot both $\boldsymbol{e}'$ and $\boldsymbol{e}$ on this coordinate system (Fig. 3), with subscripts 0 to denote this initial condition. At $t = 0$, by definition, all the vectors lie along the k-axis, so we have $c_+ + c_- = 0$ and $K_+ c_+ + K_- c_- = 0.06$. Solving the two equations we find that, at $t = 0$, $c_+ = 0.04$, $c_- = -0.04$, $K_+ c_+ = 0.047$, and $K_- c_- = 0.013$. Thus $\boldsymbol{e}'_0$ is the sum of two vectors, $\boldsymbol{c}_+$ pointing in the $+k$ direction and $\boldsymbol{c}_-$ pointing in the opposite direction. Similarly, $\boldsymbol{e}_0$ is the sum of two vectors, both pointing toward positive k, one of magnitude 0.047 and the other of magnitude 0.013. In Figure 3, the smaller dots indicate the $\boldsymbol{c}_+$ and the $K_+\boldsymbol{c}_+$ vectors. The $\boldsymbol{c}_-$ vector is exactly equal and opposite of $\boldsymbol{c}_+$, so that their total $\boldsymbol{e}'_0$ is zero. The vector $K_-\boldsymbol{c}_-$ points in the same direction as $K_+\boldsymbol{c}_+$ and their total is $\boldsymbol{e}_0$.

TABLE 1

Orbital Parameters

System	Source	Date	Planet	a (AU)	P (days)	e	ϖ (deg)	$T_{\rm peri}$ (JD − 2,450,000)	m (M_J)
47 UMa	Fischer et al. (2002)	2002 Jan	b	2.09	1089.0	0.061	171.8	3622.9	2.54
			c	3.73	2594	0.005	127	1363.5	0.76
	California and Carnegie Planet Search	2005 Feb	b	2.09	1089.0	0.061	172	3622.9	2.54
			c	3.73	2594	0.1	127	1363.5	0.76
υ And	Ford et al. (2005)	2005 Feb	c	0.825	241.32	0.258	250.2	n/a	1.943
			d	2.54	1301.0	0.279	287.9	n/a	3.943

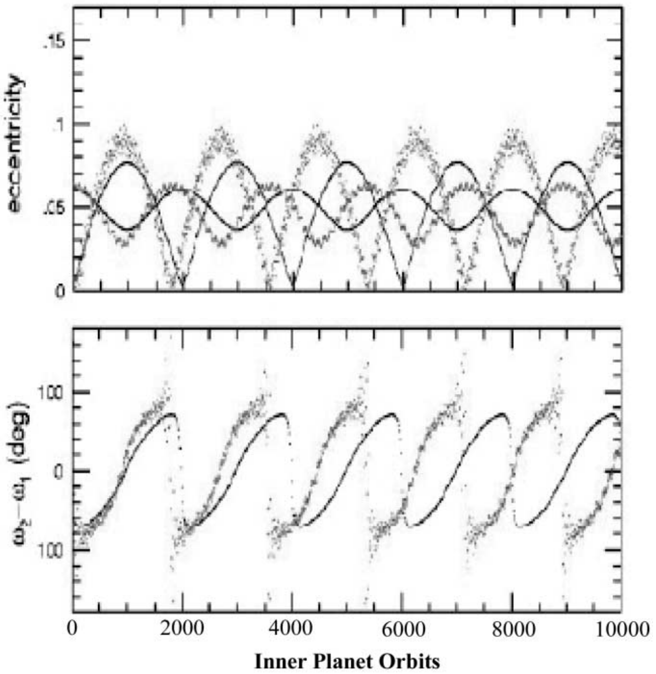

FIG. 2.—Reproduction of LCF02's Fig. 2. In this figure the black dots represent their solution to the secular theory, and grey dots represent results of an N-body integration. Although the two solutions do appear to track each other, a close examination shows several grey dots at large (close to $\pm 180°$) values of $\Delta\varpi$.

If we allow the coordinate system to rotate at the angular velocity ω_+, the vectors $\boldsymbol{c}_+$ and $K_+\boldsymbol{c}_+$ will remain fixed in the rotating coordinate system. However, $\boldsymbol{c}_-$ and $K_-\boldsymbol{c}_-$ will rotate at the rate ω_- in an inertial frame, or $\omega = \omega_- - \omega_+ \approx 10^{-3}$ yr^{-1} (or a period of 6000 yr) in the rotating frame. Hence, the total vectors $\boldsymbol{e}$ and $\boldsymbol{e}'$ will change with time as shown in Figure 4. Thus over a 6000 yr period, e' varies between 0 and 0.08, and e varies (180° out of phase) between 0.06 and 0.035, in perfect agreement with the behavior described by LCF02. Note that only a slight variation in the constant of integration c_- (which depends on initial conditions) would give either libration or circulation, depending on whether the $\boldsymbol{e}'$ circle encompasses the origin or not. With the elements used here for 47 UMa, the system is very close to the boundary between libration and circulation.

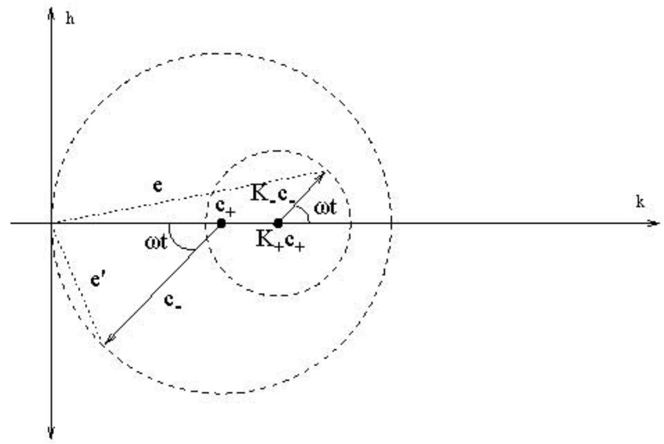

FIG. 4.—Schematic of the evolution of $\boldsymbol{e}'$ and $\boldsymbol{e}$ in a system like 47 UMa. The two vectors $\boldsymbol{c}_-$ and $K_-\boldsymbol{c}_-$ rotate about the points $k = c_+$ and $k = K_+c_+$, respectively. Note that both vectors rotate at the same rate, ω, the secular frequency. The dashed lines represent the loci of values that the two eccentricity vectors may take. The dotted lines are the eccentricity vectors. As time goes on, the magnitudes and directions of these vectors oscillate.

2.4. *Criteria for Libration*

From inspection of equation (13), or of its graphical representation (Fig. 5), we note that if either constant of integration, c_+ or c_- (determined by initial conditions), were zero, then $\boldsymbol{e}$ and $\boldsymbol{e}'$ would have fixed magnitude and point in the same direction (if K were positive) or in the opposite direction (if K were negative). Thus e and e' would be constant and $\Delta\varpi = \varpi' - \varpi$ would be fixed at 0° or 180°.

If instead of beginning at zero, one of the c values (c_- for example) were very small, then the value of $\Delta\varpi$ would librate about 0° or 180° (depending on the sign of K_+), as illustrated in

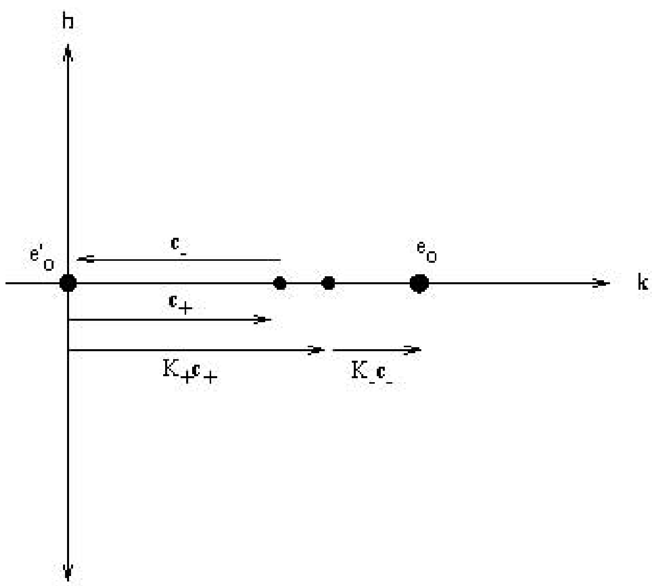

FIG. 3.—Schematic of $\boldsymbol{e}_0$, and $\boldsymbol{e}'_0$ at $t = 0$ (during apsidal alignment) in a case of near-separtrix behavior, like 47 UMa; $\boldsymbol{e}'_0$ is the sum of $\boldsymbol{c}_+$ and $\boldsymbol{c}_-$, and $\boldsymbol{e}_0$ is the sum of $K_+\boldsymbol{c}_+$ and $K_-\boldsymbol{c}_-$.

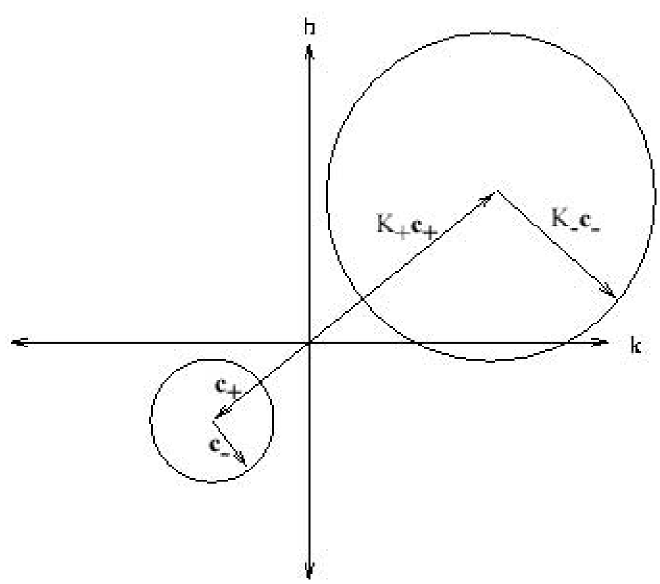

FIG. 5.—Schematic of libration. The vectors to the centers of the circles point in opposite directions. In this example, because neither circle encompasses the origin, the system is librating.

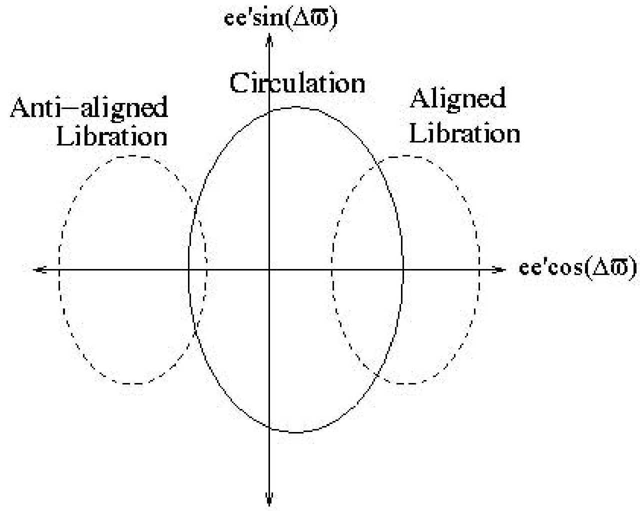

FIG. 6.—Schematic illustration of the three types of apsidal motion. Anti-alignment occurs when the solution to eq. (14) and eq. (15) lies completely in quadrants II and III. Similarly, alignment occurs when the solution lies entirely in quadrants I and IV. Circulation occurs when the solution encompasses the origin.

Figure 5. In this example, the $\boldsymbol{c}_-$ and $K_-\boldsymbol{c}_-$ vectors execute small circular "epicycles" as shown.

If $|\boldsymbol{c}_-| = |\boldsymbol{c}_+|$ or if $|K_-\boldsymbol{c}_-| = |K_+\boldsymbol{c}_+|$, then the trajectory of $\boldsymbol{e}'$ or $\boldsymbol{e}$, respectively, would periodically pass through the origin. This would be a separatrix trajectory at the boundary between libration and circulation. LCF02 used an expression for the quantity $ee' \cos(\Delta\varpi)$ as a criterion for determining the apsidal motion. In our notation the relevant equations are

$$ee' \cos(\Delta\varpi) = (K_+c_+^2 + K_-c_-^2) + (K_+ + K_-)c_+c_- \cos[(\omega_+ - \omega_-)t + (\delta_+ - \delta_-)] \tag{14}$$

and

$$ee' \sin(\Delta\varpi) = (K_- - K_+)c_+c_- \sin[(\omega_+ - \omega_-)t + (\delta_+ - \delta_-)]. \tag{15}$$

Inspection of equation (14) and Figure 4 shows that there is a critical value for the ratio of the amplitude of the oscillation of the $ee' \cos(\Delta\varpi)$ to the offset distance,

$$S = \frac{(K_+ + K_-)c_+c_-}{K_+c_+^2 + K_-c_-^2}. \tag{16}$$

If $|S| > 1$, $\Delta\varpi$ circulates. If $|S| < 1$, $\Delta\varpi$ librates. Moreover, if the offset, the denominator in equation (16), is positive, $\Delta\varpi$ librates about 0°. If it is negative, libration is about 180°. If $|S| = 1$, the trajectory lies on the separatrix.

As we have seen, for the initial conditions used by LCF02, the behavior is very close to the separatrix. Moreover, LCF02 noted that for 47 UMa, $|S| < 1$ and inferred that, at least for the orbital elements and secular theory that they used, the system librates. The trajectory in a Cartesian coordinate system $\{ee' \cos(\Delta\varpi), ee' \sin(\Delta\varpi)\}$ is therefore an ellipse, like the examples shown in Figure 6.

2.5. *What Is a Resonance?*

A resonance occurs when natural frequencies of a system are commensurate. Consider the apsidal precession frequency of an orbiting planet. The planet may precess, while maintaining a constant eccentricity, due, for example, to perturbations by the oblateness of the central body. Similar perturbations of the same mathematical form will result from other orbiters on circular orbits, from general relativity, from rings of small orbiters, etc. These combined effects define the "natural" precession frequency. Now, if the orbiter is affected by another planet that is not on a circular orbit, the perturbations are more complicated (e.g., § 2.1). However, the term in the disturbing function that does not contain the perturber's eccentricity (i.e., the first term in eqs. [1] or [2], depending on whether the perturber is interior or exterior) has the same functional form as any of the other effects (e.g., oblateness) that contribute to the natural precession. Hence, in terms of the dynamics, the effect of this term must be included in the natural precession frequency.

For the system of two planets introduced in § 2.1, the natural precession rates are thus obtained by ignoring the second terms in the disturbing functions. In that case, if we used only the first terms, equation (7) would give simple precession of the planets at the rates A and D, respectively. Thus A and D are, by definition, the natural precession rates for this system.

A secular resonance occurs if $A \approx D$. In order to evaluate the effects of the resonance, let $A = D$ in our secular theory developed in § 2.1. In this case, from equation (11),

$$K_\pm = \pm\sqrt{B/C}, \tag{17}$$

and, from equation (10),

$$\omega_\pm = D \mp \sqrt{CB}. \tag{18}$$

From equations (14) and (17), we have

$$ee' \cos(\Delta\varpi) = K_+c_+^2 + K_-c_-^2 = K_-(c_-^2 - c_+^2), \tag{19}$$

which is constant. From equation (15) we have

$$ee' \sin(\Delta\varpi) = 2K_-c_+c_- \sin[\sqrt{CB}t + (\delta_+ - \delta_-)]. \tag{20}$$

So in resonance, the maximum amplitude of libration of $\Delta\varpi$ is (cf. Fig. 6)

$$\Delta\varpi_{\max} = \tan^{-1}\left(\frac{2c_+c_-}{c_-^2 - c_+^2}\right). \tag{21}$$

The relationship between libration and resonance has not always been clear in the literature (e.g., Barnes & Quinn 2004; LCF02; Malhotra 2002). These papers use the term secular resonance throughout to mean libration. Kinoshita & Nakai (2001) stated that a secular resonance occurs when two eigenfrequencies (ω_+ and ω_- in the two-planet case) are equal. According to equation (10), the Kinoshita & Nakai (2001) definition requires $(D - A)^2 + 4CB = 0$. Our definitions of these constants requires that they all be positive definite. Therefore the Kinoshita & Nakai (2001) definition of secular resonance can never be met for a two-planet system. The correct condition for resonance is when the natural frequencies A and D are equal.

Next we demonstrate behavior of the eccentricities when this resonance condition ($A = D$) is met. Suppose that we let $t = 0$, when $e = e_0$, $e' = e'_0$ and $\varpi'_0 = \varpi_0 = 0$. Using equation (17) we obtain

$$c_+ + c_- = e'_0,$$
$$K_+c_+ + K_-c_- = e_0. \tag{22}$$

Therefore,

$$c_+ = \frac{e'_0}{2} + \frac{e_0}{2K_+},$$
$$c_- = \frac{e'_0}{2} - \frac{e_0}{2K_+}. \tag{23}$$

We next show that even if e_0 and e'_0 are small, at least one eccentricity can become very large, given the condition $A = D$.

The behavior depends critically on the value of K_+ and may be divided into three cases.

Case 1: $|K_+| \gg e_0/e'_0$.—Assuming $e_0 \sim e'_0$, this condition can only be met if $m'/m \gg 1$, according to equations (7) and (17). In this case, $c_+ \approx c_-$. In other words, the two vectors that compose $\boldsymbol{e}'_0$ in (k, h) space are equal. Similarly, $e_0 = K_+c_+ + K_-c_-$, or (because $K_- = -K_+$), $e_0 = K_+(c_+ - c_-)$, i.e., $\boldsymbol{e}_0$ is the sum of two nearly equal and opposite vectors with large amplitudes. As the directions of $\boldsymbol{c}_+$ and $\boldsymbol{c}_-$ change at rates ω_+ and ω_-, the value of e will increase to $K_+(c_+ + c_-)$. Therefore, as the mass ratio between the planets grows, the eccentricity of the smaller planet may grow to arbitrarily large values. This case, with $m' \gg m$ was considered by Greenberg (1975), who showed that $A-D$ appears in the denominator of the solution, driving e toward very large values when natural precession rates A and D are nearly equal.

Case 2: $|K_+| \ll e_0/e'_0$, or $m/m' \gg 1$.—In this case, c_+ and c_- are nearly equal and opposite according to equation (23). From equation (22), a finite e_0 implies $|c_+|$ and $|c_-|$ must be large. As K_+ is small, this suggests that as $\boldsymbol{c}_+$ and $\boldsymbol{c}_-$ rotate into a parallel configuration $e = c_+ + c_-$ becomes large.

Case 3: $|K_+| \sim 1$.—In this case there is no substantial increase in either $\boldsymbol{ee}$ or e', which remain close to e_0 and e'_0 for all time, respectively.

In summary, for two planets a secular resonance occurs if the natural precession frequencies A and D are nearly equal. In addition, if initial eccentricities are comparable to one another, we find that resonant pumping of an eccentricity to a high value requires the planets' masses to be significantly different from one another. The identification of libration as synonymous with resonance, common in the literature, is not correct. In fact, libration is possible even very far from resonance. One example is the alignment of semimajor axes of Saturn's satellites Titan and Rhea, which are not in resonance (Greenberg 1975). However, secular resonance behavior does tend to include libration, so libration may be a useful indication, if not proof, of resonance. Finally, we emphasize that resonance is not characterized by nearly equal eigenfrequencies, but rather by nearly equal natural frequencies in a system with two planets.

3. COMPARING NUMERICAL AND ANALYTICAL RESULTS

We have recomputed the behavior of eccentricities and longitudes of pericenter for the 47 UMa system, using the 2002 best estimate of elements (as in § 2 and in LCF02) as shown in Figure 7. As in the version published by LCF02 (reproduced in Fig. 2), we show the results of both the analytical secular theory and a numerical integration performed with the code MERCURY6 (Chambers 1999).

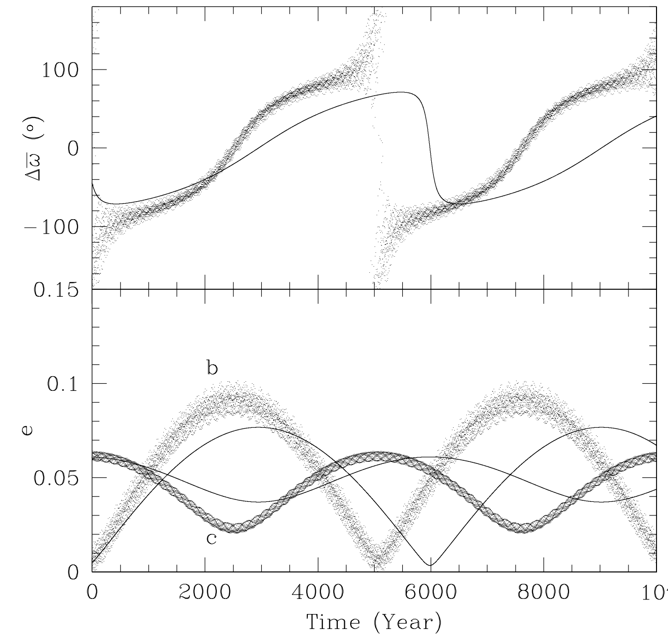

Fig. 7.—Evolution of the astrocentric longitudes of periastron (*top*) and eccentricities (*bottom*) for the Fischer et al. (2002) configuration of 47 UMa presented in Table 1. The solid lines represent the analytic solution; the points, a numerical integration (sampled yearly). Secular theory predicts aligned libration, but the numerical simulation is clearly circulating. This figure demonstrates a breakdown in secular theory when applied to extrasolar planetary systems. The apparent high frequency scatter while the system passes through $|\Delta\varpi| = 180°$ is an artifact of plotting in astrocentric coordinates. We plot the numerical results in this coordinate system as secular theory provides orbital elements in this system. Although not shown, in Jacobi elements, the orbital elements have less scatter, and there are no points with $|\Delta\varpi| < 90°$ when $e_b \approx 0$.

First the analytical results are essentially identical. They show e_c dropping to near zero, as is characteristic of behavior near the separatrix. They show libration of $\Delta\varpi$ with an amplitude of nearly 90° and a rapid change in $\Delta\varpi$ as e_c passes near zero. Second, in both our results and those of LCF02, the numerical behavior gives a period about 15% shorter than the analytical case, and variations in the eccentricities have larger amplitudes. These differences between analytical and numerical results reflect the assumptions of the analytical work.

We sample the numerical simulation of the system in 1 yr intervals, providing more data points than were in LCF02. This extra information shows that $\Delta\varpi$ actually circulates rather than librates, although there is substantial short-term scatter in the values of e, e', and $\Delta\varpi$.

The behavior can be better understood by considering the motion in a polar plot of $ee' \cos \Delta\varpi$ versus $ee' \sin \Delta\varpi$ (Fig. 8). Here we see confirmation that the analytical result gives libration, albeit near the separatrix. However, the numerical result clearly shows circulation, still near the separatrix, but now embracing the origin. Note that the short-term scatter in the trajectory can produce enormous changes in $\Delta\varpi$, but these only occur near the origin where e' is near zero. At that point, with a nearly circular orbit, very small changes in the orbit can wildly change the orientation of the major axis, which is not well-defined at $e' = 0$.

The circulation of $\Delta\varpi$ demonstrated by the numerical results is contrary to the results reported by LCF02, who said that the numerical results confirmed the libration predicted by the

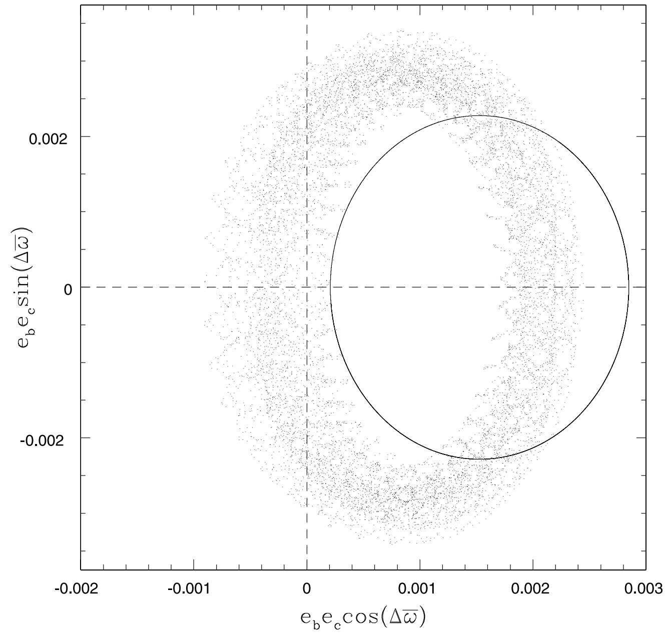

FIG. 8.—Polar plot of the solutions to eqs. (14) and (15). The secular solution (*solid line*) does not encompass the origin and is therefore librating. Since the ellipse lies entirely in the positive $ee' \cos(\Delta\varpi)$ direction, the system is in an aligned configuration. Conversely, the numerical results (*dots*), do surround the origin, and therefore the system circulates. Compared to the width of these ovals, their distances from the origin are small. This figure shows how a small quantitative difference can produce such a drastic qualitative difference. In addition, the secular solution undergoes a large librational amplitude. The current best fit to this system, therefore, lies close to the separatrix, between aligned libration and circulation. The numerical model appears to pass through the origin, but this again is an artifact of using astrocentric coordinates. In Jacobi coordinates the numerical oval passes completely to the left of the origin.

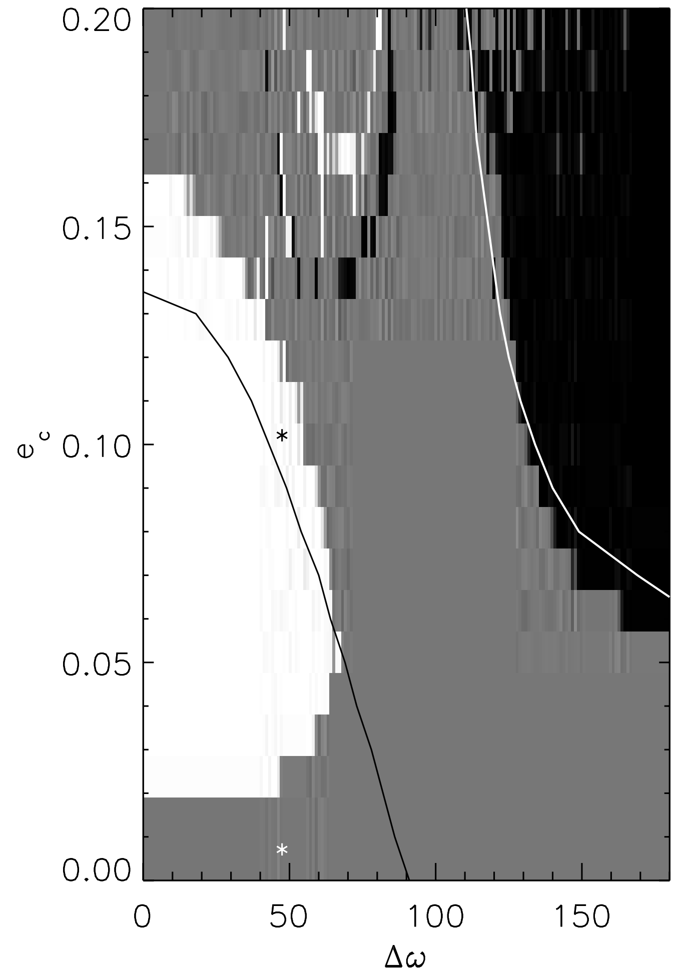

FIG. 9.—Distribution of apsidal motion as predicted from secular theory and resulting from numerical integrations. Secular theory predicts aligned libration in the region bounded by the black line, antialigned libration by the white line, and circulation elsewhere. White regions represent numerical trials which underwent aligned libration; black, antialigned; and grey, circulation. The lines do approximate the shaded regions, but the "old" system ($e_c = 0.005$, $\Delta\varpi = 45°$, *white asterisk*) is incorrectly labeled as aligned libration by secular theory. A revision to the elements changed e_c to 0.1. The black asterisk marks this "new" location. Now the problem is reversed; secular theory predicts circulation, but the system is actually undergoing aligned libration.

analytical secular theory. Had they plotted their numerical results on a polar plot or plotted points more closely spaced in time, they might have seen that the system circulates rather than librates. Certainly their statement that the system lies "deep inside" the libration zone is wrong based on either the analytical theory or the numerical results. Either way, the trajectory lies close (the trajectory's proximity to the origin is much less than the size of the trajectory in these parameters) to the separatrix, not deep inside the libration zone. Moreover, the system is so close to the separatrix that any difference is comparable to the magnitude of short-period effects and to the difference between analytical and numerical integration.

Our result shows the limitations of the analytical theory, with its various assumptions (e.g., very small eccentricities). The analytical results are qualitatively correct and useful. However, numerical integration gives more accurate results, which can be significantly different from the analytical results. In this case, it means the difference between libration and circulation.

LCF02 introduced a mapping of apsidal motion in initial $\Delta\varpi$ versus initial e' space, where libration or circulation are expected to follow from initial conditions, with boundary lines separating the different domains. The boundaries were based on the analytical secular theory and are shown in Figure 9. In this figure the solid lines represent the separatrices as determined by secular theory. In addition we have used the numerical simulations to more accurately define the domain of libration and circulation (as shown by the shading).

For the numerical study, the parameter space of Figure 9 is divided into bins of 0.01 in $e'_0 = e_c$ and 1° in $\Delta\varpi$. In each bin we solve the secular solution and run an *N*-body simulation for 10^4 yr (nearly two secular periods; cf. Fig. 7). For numerical integrations we sampled $\Delta\varpi$ once per (Earth) year and determined if the system ever came within 20° of either parallel or antiparallel alignment.

The lower left corner, bounded by the black line, represents parameter space that secular theory predicts to be aligned libration. The top right corner, bounded by the white line, is antialigned libration. Secular theory predicts the intervening region to be circulating. The colors represent the results of numerical simulations. White regions underwent aligned libration, grey circulated, and black were antialigned. The shading shows that the boundaries between circulation and libration from the numerical experiments are different from the boundaries from the analytical solution. A prominent disagreement, for example, is at low values of e' with $\Delta\varpi$ between 90° and 0°. While the analytical theory predicts libration there, the numerical experiments show circulation. Consequently, for the initial conditions of 47 UMa assumed by LCF02 (the white asterisk at $e_c = 0.005, \Delta\varpi = 47°$), the system circulates, contrary to the libration predicted by secular theory.

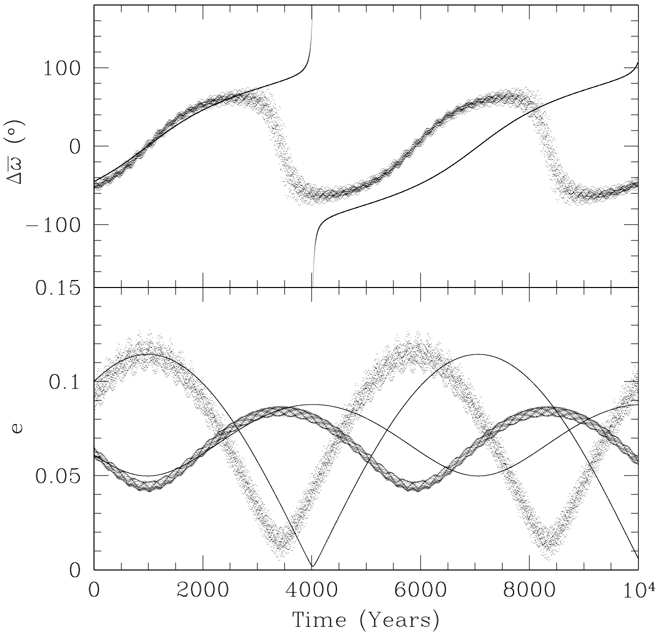

FIG. 10.—Evolution of the new orbital elements of 47 UMa. In this case, secular theory (*solid lines*) predicts circulation, whereas the system is clearly librating, based on the results of a numerical integration (*dots*).

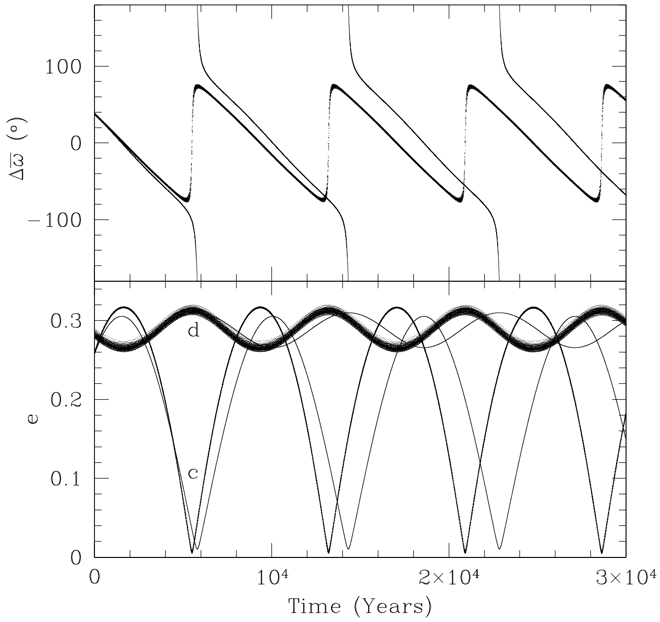

FIG. 12.—Secular evolution of the outer two planets of υ And. In this analysis, the inner planet has been ignored. As in 47 UMa we see that the analytical solution (*thin lines*, *longer period*) does not match the apsidal motion of the numerical solution (*thick lines*, *shorter period*). In this case, however, the analytical solution does a good job of matching the eccentricity evolution and the secular period.

Next we consider the more recently improved orbital elements, shown in Table 1 and by the black asterisk in Figure 9. At this point in the map, the analytical criterion predicts circulation, while the numerical simulation predicts libration, each the opposite of the results for the earlier reported elements. Thus while LCF02 were incorrect in reporting that numerical integration gave libration with earlier elements, with the new elements, numerical results do give libration after all. With either set of elements, the system is close to the separatrix.

The numerical integration of the current best fit to 47 UMa is shown in Figures 10 and 11. The libration near the separatrix is evident. Note however, that the uncertainties in the orbital parameters are still substantial, 0.1 for e' and 45° for $\Delta\varpi$. With these uncertainties, and with the system so close to the separatrix, it remains undetermined whether the 47 UMa system is actually in a state of circulation or libration.

Another system that similarly lies near the secular separatrix is υ And (Butler et al. 1999; Ford et al. 2005). Using the current best-fit estimates for orbital elements (see Table 1), while again

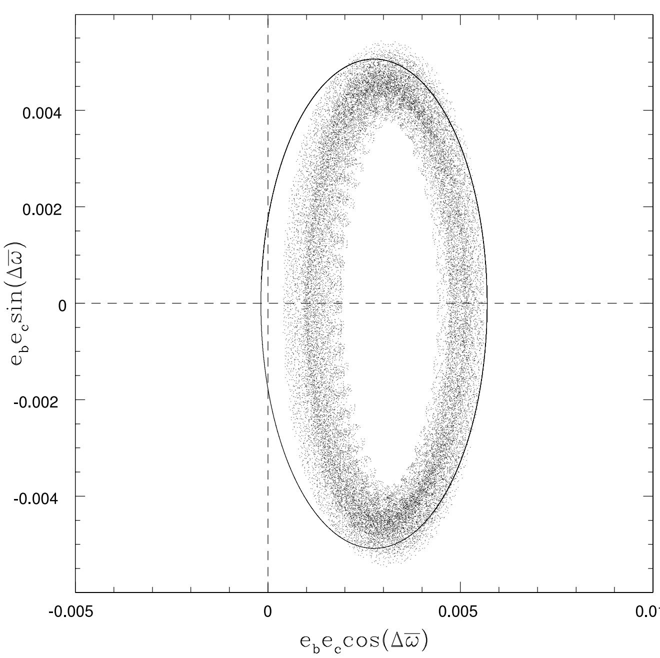

FIG. 11.—Polar plot for the new orbital elements of 47 UMa. As expected from Fig. 10, the analytical solution encompasses the origin, but the numerical simulation lies entirely in quadrants I and IV.

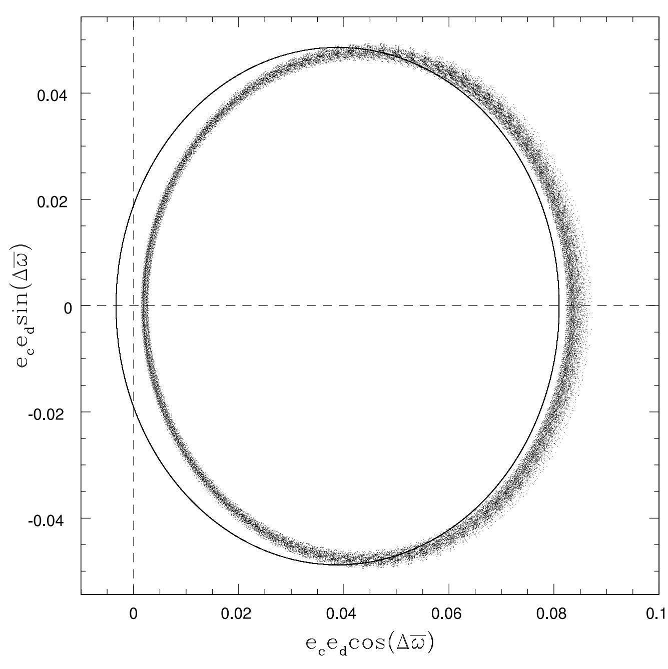

FIG. 13.—Polar plot for υ And. This system appears to lie even closer to the separatrix than 47 UMa. In this system the analytical solution (*line*) encompasses the origin and predicts circulation, but the numerical simulation (*points*) shows the system is actually librating.

recognizing that the observations admit some uncertainty in the values, we plot the behavior in Figures 12 and 13, similar to Figures 10 and 11 for 47 UMa, above. Again the behavior is very close to the separatrix, although in this case it is the interior planet's eccentricity, e, that passes close to zero, rather than e', as in the case of 47 UMa.

In the υ And case, as in 47 UMa, there is a distinct difference between the results from the numerical solution and the results from the analytical secular theory. As seen in Figure 13, the secular theory gives circulation and the analytical theory gives libration. Again this qualitative difference is made possible by the fact that the system is close to the separatrix. Based on the larger orbital eccentricities in this case (e and e' both exceed 0.3) than in the 47 UMa case (neither exceeding 0.1), one might have expected much greater discrepancy between the analytical and numerical results. However, in comparing Figures 11 and 13, we find the differences are comparable for both these planetary systems. We conclude that for systems near a secular separatrix, first-order secular theory should not be used to determine the apsidal motion. Only direct N-body integrations give the true motion.

4. ORIGINS OF NEAR-SEPARATRIX MOTION

We have seen that two planetary systems, 47 UMa as well as the previously recognized υ And, display secular orbital behavior very close to the separatrix between libration and circulation of the orientations of the major axes of the planets. Near-separatrix behavior is characterized by one of the planet's orbital eccentricities periodically passing near a value of zero. More specifically, in the polar plots, the width of the trajectory is much larger than the closest approach to the origin.

The υ And system inspires consideration about how such a state might be created. If both planets were initially on near-circular orbits, and then one of them was suddenly given an eccentricity (for example, by an impulsive force), the initial conditions would be set up for a near-separatrix trajectory of the system in orbital element space, as discussed in § 2. Malhotra (2002) suggested that such an impulse in the past might explain the state of υ And, with the impulse provided by scattering from another planet. This idea was pursued further by Ford et al. (2005), who demonstrated the process with a hypothetical system of three planets initially on circular orbits, whose evolution they studied with a numerical simulation. The system was started with the two outer planets on orbits close enough that they quickly began to pump one another's orbital eccentricities, until in a short time one escaped the system and the other had acquired a substantial eccentricity. The inner planet was still on a near-circular orbit. Thus subsequent motion followed the near-separatrix behavior exhibited by υ And.

The agreement with the observed system's behavior makes this a compelling model for the origin of near-separatrix trajectories. However, the initial conditions assumed in that scenario need to be considered more carefully. The initial system with all three planets on circular orbits must have evolved into that state by the planets either growing in place or migrating into those orbits, or both. Remaining to be explained is how, during that evolution, prior to the initial state assumed by Ford et al. (2005), the system avoided already having eccentricities pumped up. It is not clear that the assumed system of circular orbits could ever have formed.

Is it possible instead that the near-separatrix state of υ And is simply a matter of chance initial conditions? To address the plausibility of that explanation, consider a hypothetical system similar to υ And in terms of masses and semimajor axes. In general, according to secular theory, the major axes will line up periodically. We can define such an instant as time $t = 0$, with the "initial" condition given by the state of the system. For example, suppose that the inner planet has an orbital eccentricity of $e = 0.3$. This example value happens to be the value of e when the major axes are aligned in the real υ And system (Fig. 12), but in fact the choice of this particular numerical value is not critical. The following argument will hold for any reasonable eccentricity value in the typical range for extrasolar planets. If the outer planet's eccentricity e had exactly the "ideal" initial value (which also happens to be 0.3), the system would lie exactly on the separatrix and the value of e would periodically plunge to exactly zero. However, the secular solution shows that the initial value of e could differ from this ideal value over a range of about 5% and still give behavior at least as close to the separatrix as that of the actual υ And system.

Thus, roughly speaking, the probability of random initial conditions giving near separatrix behavior is at least a few percent. A more rigorous analysis of this probability would use action-angle variables (for which the volume of a region of phase space is conserved under the time evolution of a Hamiltonian system) rather than Keplerian elements. However for our purposes, this rough estimate is adequate. Given that 18 multiplanet systems are known (including our solar system), it is not surprising to find one of them by chance as close to the separatrix as υ And is. Thus υ And does not by itself require a special model to explain its behavior.

The story changes somewhat now that we have also found the current best fit to 47 UMa to be near the separatrix. In fact, the probability that 47 UMa would have random initial conditions that put it so close to the separatrix would also be a few percent. Thus, while the state of υ And by itself does not indicate any need for an explanation, the fact that two systems have this condition does suggest that there is some process that tends to favor it. Furthermore, we have recently noted that at least two pairs of extrasolar planets that are in mean motion resonances (55 Cnc and HD 128311) also display behavior of $\Delta\varpi$ that is very near the separatrix between libration and circulation (Greenberg & Barnes 2005). Therefore, four out of the 24 known pairs of adjacent giant-planet orbits (including our own solar system) display near-separatrix behavior, while only about 1 would be expected by chance. It appears that some process favors this condition, and further investigation may help provide insight into the formation and evolution of planetary systems.

Finally, we emphasize that the surprisingly common phenomenon of near-separatrix behavior should no longer be confused with secular resonances or with libration. Neither 47 UMa or υ And is in or near a secular resonance, and it is uncertain whether either is librating. What makes them remarkable is how close each system is to the boundary between libration and circulation.

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