Immersed Interface Methods for Fluid Dynamics Problems

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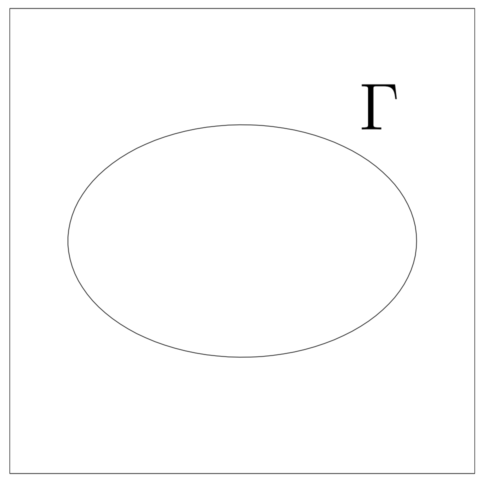
Incompressible Navier-Stokes

$$
\begin{aligned}
u_t + (u \cdot \nabla)u + \nabla p &= \mu \nabla^2 u + f \\
\nabla \cdot u &= 0
\end{aligned}
$$



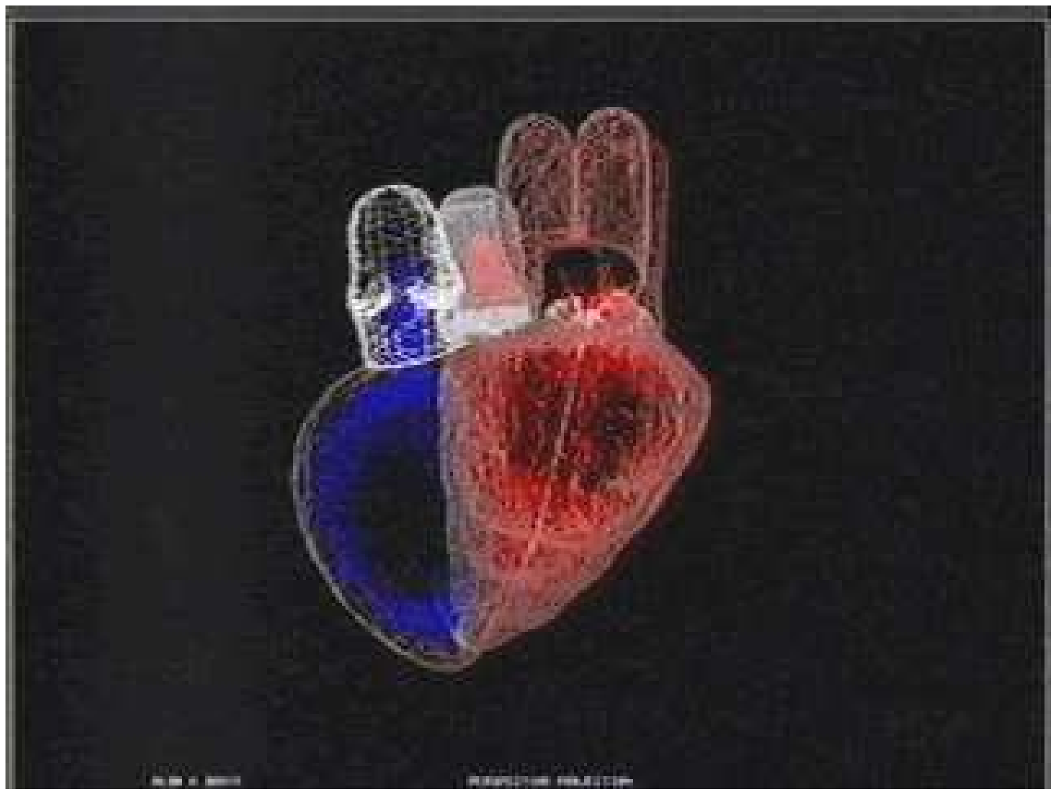
An immersed elastic membrane Γ exerts a singular force on the fluid,

$$
f(x, y) = \int_\Gamma F(s)\, \delta(x - X(s))\, \delta(y - Y(s))\, ds,
$$

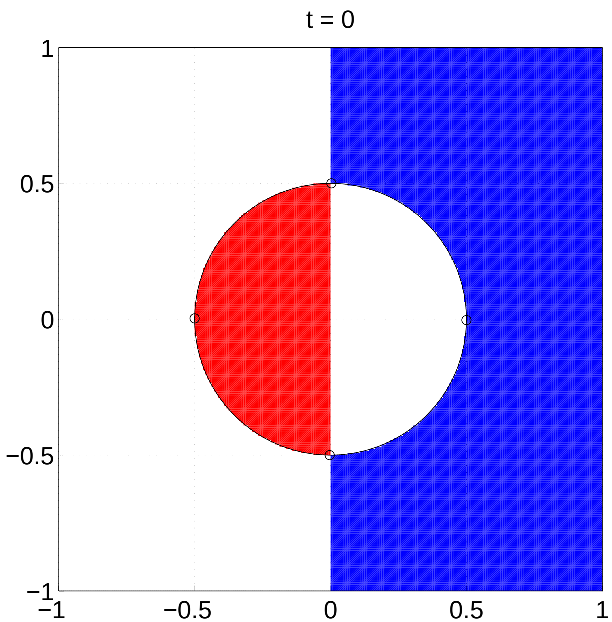
and moves with the fluid.

Peskin's Heart Model

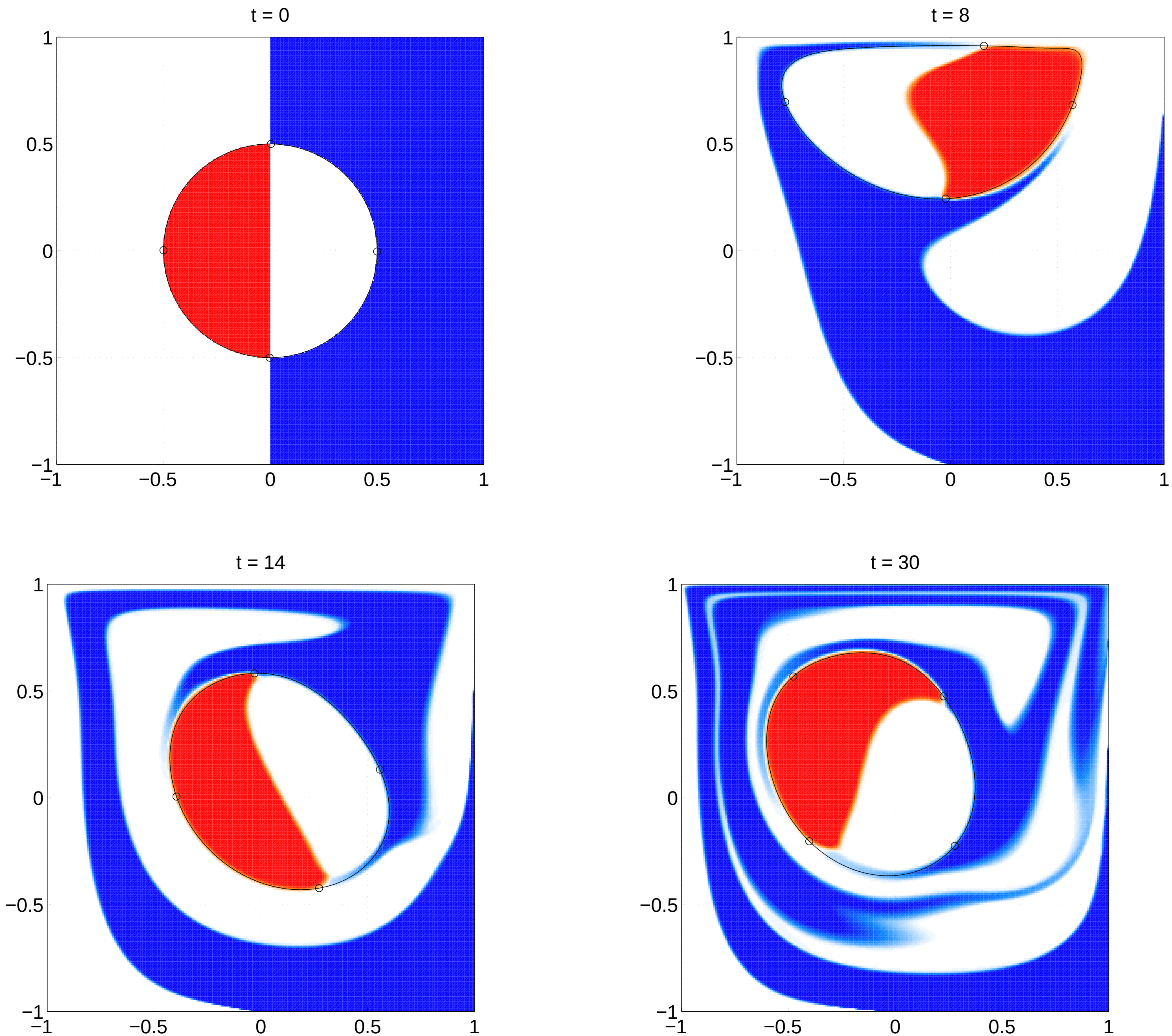
Originally developed to model blood flow in a beating heart and the operation of artificial heart valves.



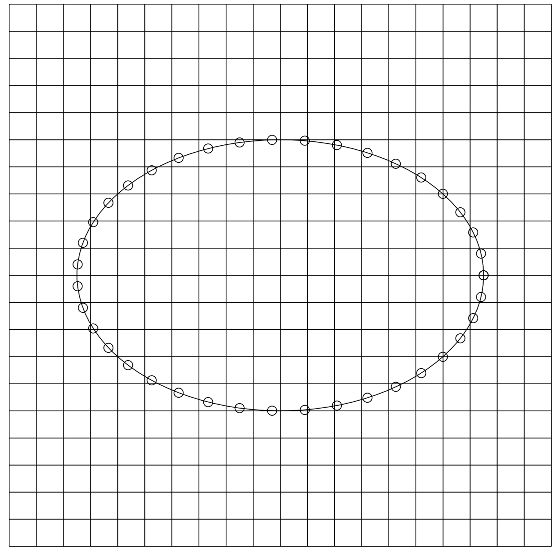
Balloon in a driven cavity



Balloon in a driven cavity



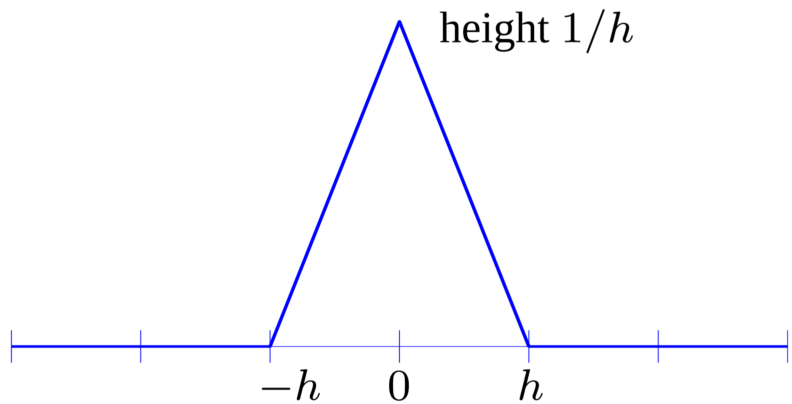
Peskin's Immersed Boundary Method



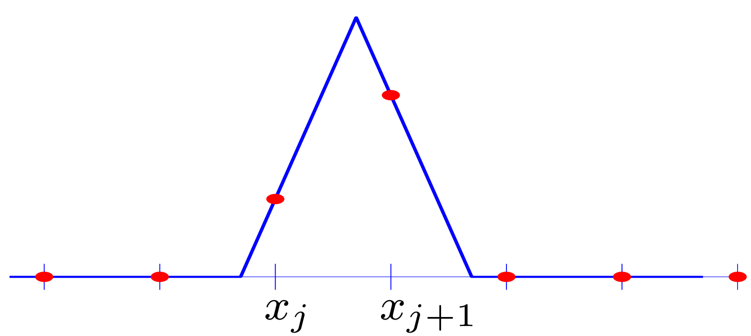
- Membrane represented by discrete control points X_k^n.
- Calculate force strength F_k^n at each control point.
- Use discrete delta function to spread forces to nearby Cartesian grid points, yielding nonzero f_{ij} at points near the interface.
- Advance the fluid equations on the uniform grid.
- Interpolate resulting velocity field u_{ij}^{n+1} to control points to obtain U_k^{n+1}.
- Move control points by $X_k^{n+1} = X_k^n + \Delta t\, U_k^{n+1}$.
- Implicit or semi-implicit approach may be needed for stability.

Discrete delta function in 1D

Example: Hat function



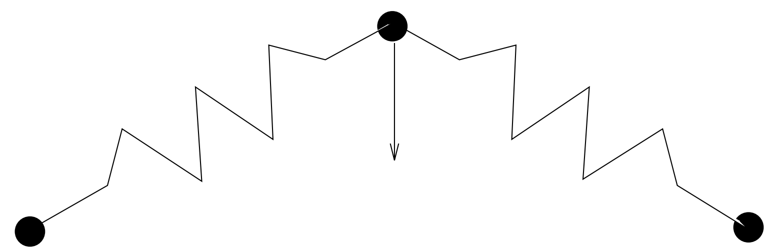
Singular force $F\delta(x-\alpha) \approx F d_h(x_i - \alpha)$ on the grid.
This is nonzero at only two points $(x_j < \alpha < j_{j+1})$:



Spring model of forces

The force $\vec{F}_k$ at $\vec{X}_k$ is computed based on the shape of the boundary.

Example: Spring model



$$\vec{F}_k = \sigma_{k+1/2}(\vec{X}_{k+1} - \vec{X}_k) - \sigma_{k-1/2}(\vec{X}_k - \vec{X}_{k-1}).$$

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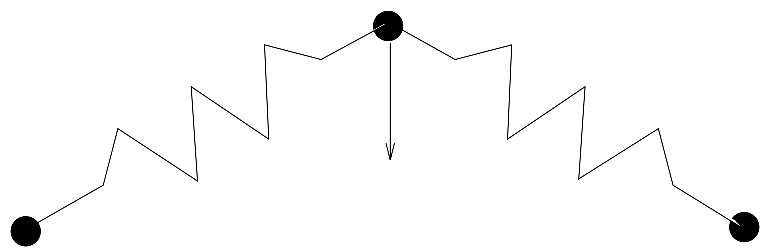


$$\vec{F}_k = \sigma_{k+1/2}(\vec{X}_{k+1} - \vec{X}_k) - \sigma_{k-1/2}(\vec{X}_k - \vec{X}_{k-1}).$$

For $\vec{X}(s)$ parameterized by unstretched length,

$$f(s,t) = \frac{\partial}{\partial s}(T(s,t)\tau(s,t)),$$

where

$$T(s,t) = T_0\left(\left|\frac{\partial X(s,t)}{\partial s}\right| - 1\right).$$

Jump conditions

With mass density $m(s)$:

$$m(s)X_{tt}(s,t) = f(s,t) - [\![p]\!]\vec{n} + \mu\left[\frac{\partial u}{\partial n}\right].$$

Massless membrane: $m(s) = 0$

$$\begin{aligned} f &= \text{elastic force (computed from } X^n\text{)} \\ &= f_n\vec{n} + f_\tau\tau \\ [\![p]\!] &= f_n \\ \mu\left[\frac{\partial u}{\partial n}\right] &= -f_\tau\tau \end{aligned}$$

Projection Method (one form)

$$\begin{aligned} u_t + (u \cdot \nabla)u + \nabla p &= \mu \nabla^2 u + f \\ \nabla \cdot u &= 0 \end{aligned}$$

$\boxed{1}$ $U^n \longrightarrow U^*$ by solving

$$u_t + (u \cdot \nabla)u = \mu \nabla^2 u + f$$

$\boxed{2}$ $U^* \longrightarrow U^{n+1}$ by solving

$$u_t + \nabla p = 0$$

and requiring $\nabla \cdot U^{n+1} = 0$:

$$\frac{U^{n+1} - U^*}{\Delta t} + \nabla p = 0$$

$$\Longrightarrow \quad \Delta t \, \nabla^2 p = \nabla \cdot U^*$$

$\boxed{1}\ U^n \rightarrow U^*$ by solving $u_t + (u \cdot \nabla)u = \mu \nabla^2 u + f$

$\boxed{2}\ U^* \rightarrow U^{n+1}$ by solving $\Delta t\, \nabla^2 p = \nabla \cdot U^*$

True solution:

- p should be discontinuous across Γ
- u should be continuous but not smooth

Numerical solution:

- Singular source in $\boxed{1}$ leads to "delta function" in U^*
- $\nabla \cdot U^*$ gives "dipole source" for $\nabla^2 p$
- Results in "discontinuity" in p.

Immersed Interface Approach

$$
\begin{aligned}
u_t + (u \cdot \nabla) u + \nabla p &= \mu \nabla^2 u + f \\
\nabla \cdot u &= 0
\end{aligned}
$$

$\boxed{1}$ $U^n \longrightarrow U^*$ by solving

$$u_t + (u \cdot \nabla) u = \mu \nabla^2 u + f_\tau$$

$\boxed{2}$ $U^* \longrightarrow U^{n+1}$ by solving

$$u_t + \nabla p = f_n$$

and requiring $\nabla \cdot U^{n+1} = 0$:

$$\frac{U^{n+1} - U^*}{\Delta t} + \nabla p = f_n$$

$$\Longrightarrow \quad \Delta t \, \nabla^2 p = \nabla \cdot U^* + \Delta t \, \nabla \cdot f_n$$

$\boxed{1}$ $U^n \longrightarrow U^*$ by solving $u_t + (u \cdot \nabla)u = \mu \nabla^2 u + f_\tau$

$\boxed{2}$ $U^* \longrightarrow U^{n+1}$ by solving $\Delta t\, \nabla^2 p = \nabla \cdot U^* + \Delta t\, \nabla \cdot f_n$

Numerical solution:

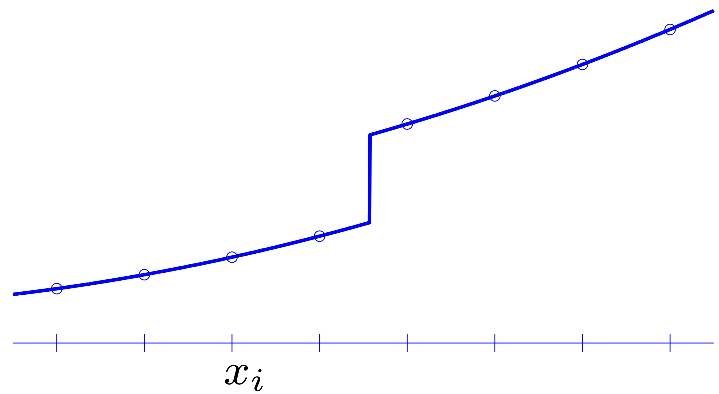
- Singular source in $\boxed{1}$ is tangential to interface, so u remains bounded
- $\nabla \cdot f_n$ gives correct dipole source for $\nabla^2 p$
- Use jump conditions on p and $\partial p / \partial n$ while solving

$$\Delta t\, \nabla^2 p = \nabla \cdot U^*$$

using an immersed interface method.

Simple 1D example: $p_{xx} = f(x)$ with boundary conditions and jump condition $[\![p]\!] = c$ at $x = \alpha$.

Or: $p_{xx} = f(x) + c\delta'(x - \alpha)$.



Want to set $p_{xx}(x_i) = f(x_i)$.

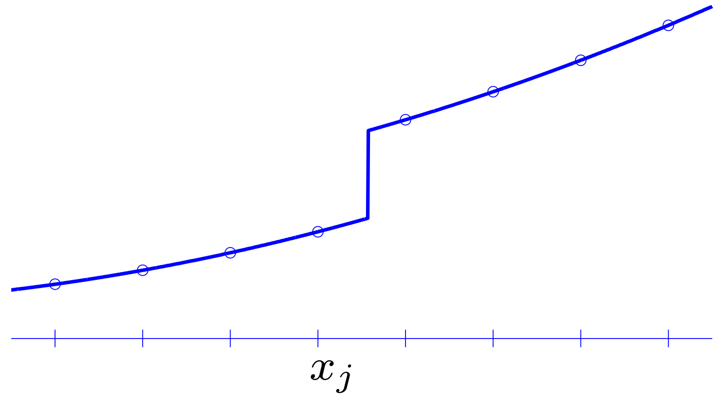
$$p(x_{i-1}) = p(x_i) - hp_x(x_i) + \frac{1}{2}h^2 p_{xx}(x_i) - \cdots$$
$$p(x_{i+1}) = p(x_i) + hp_x(x_i) + \frac{1}{2}h^2 p_{xx}(x_i) + \cdots$$

$$\Longrightarrow \qquad p_{xx}(x_i) \approx \frac{p(x_{i-1}) - 2p(x_i) + p(x_{i+1})}{h^2}$$

So that

$$\frac{p_{i-1} - 2p_i + p_{i+1}}{h^2} = f_i$$

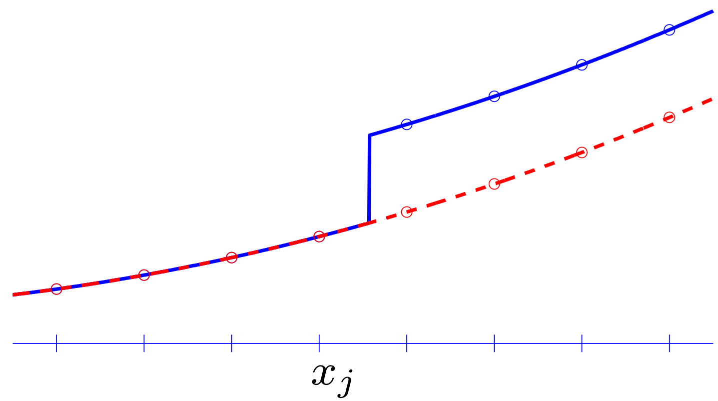
$$p_{xx} = f(x) + c\delta'(x - \alpha).$$



Want to set $p_{xx}(x_j) = f(x_j)$.

$$p(x_{j-1}) = p(x_j) - hp_x(x_j) + \frac{1}{2}h^2 p_{xx}(x_j) - \cdots$$

$$p_{xx} = f(x) + c\delta'(x-\alpha).$$



Want to set $p_{xx}(x_j) = f(x_j)$.

$$p(x_{j-1}) = p(x_j) - hp_x(x_j) + \tfrac{1}{2}h^2 p_{xx}(x_j) - \cdots$$
$$p(x_{j+1}) = \big(p(x_j) + hp_x(x_j) + \tfrac{1}{2}h^2 p_{xx}(x_j) + \cdots\big) + c$$

$$\Longrightarrow \qquad p_{xx}(x_j) \approx \frac{p(x_{j-1}) - 2p(x_j) + p(x_{j+1})}{h^2} - \frac{c}{h^2}$$

So that

$$\frac{p_{j-1} - 2p_j + p_{j+1}}{h^2} = f_j + \frac{c}{h^2}$$

and similarly

$$\frac{p_j - 2p_{j+1} + p_{j+2}}{h^2} = f_{j+1} - \frac{c}{h^2}$$

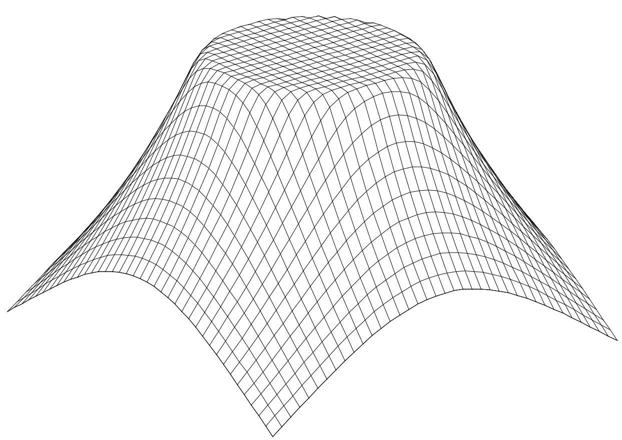
Example: $u_{xx} + u_{yy} = \int_\Gamma \delta(x - X(s))\, \delta(y - Y(s))\, ds$

We use the Dirichlet boundary condition which is determined from the exact solution

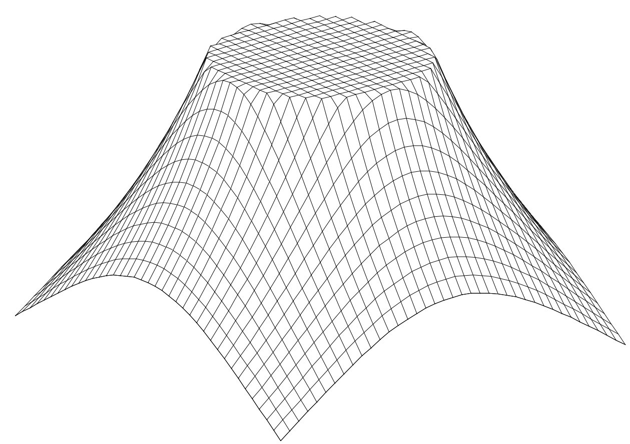
$$u(x,y) = \begin{cases} 1 & \text{if } r \le 0.5 \\ 1 + \log(2r) & \text{if } r > 0.5 \end{cases}$$

where $r = \sqrt{x^2 + y^2}$.

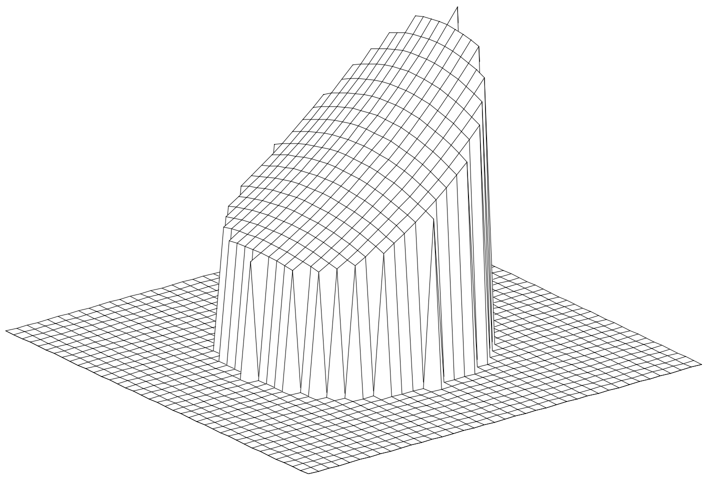
Using $d_h(x)$:

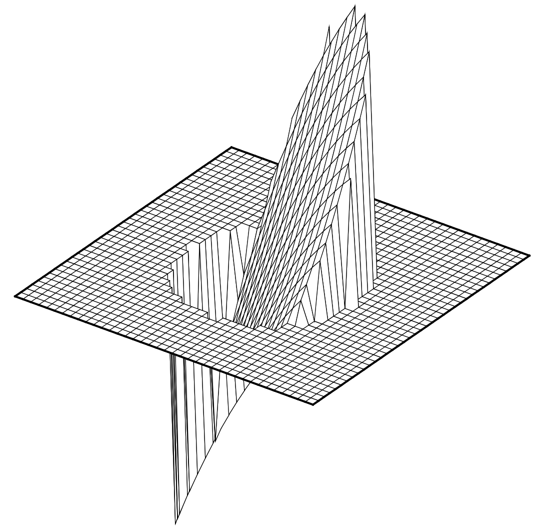


Using jump conditions:



Example with discontinuity in solution

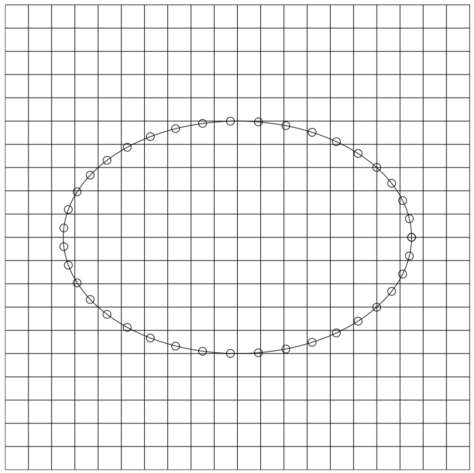
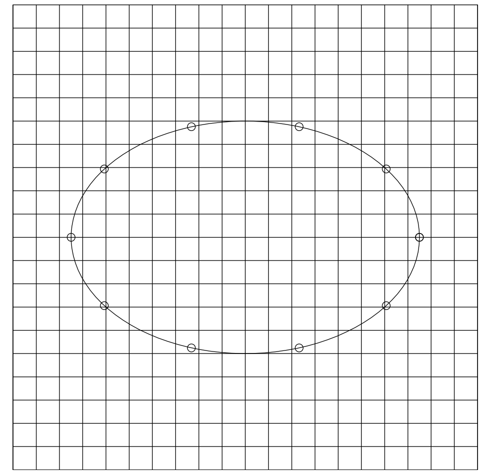




n	global error	ratio	local error	ratio
20	4.37883×10^{-4}		2.99215×10^{-2}	
40	1.07887×10^{-4}	4.0587	1.52546×10^{-2}	1.9615
80	2.77752×10^{-5}	3.8843	7.70114×10^{-3}	1.9808
160	7.49907×10^{-6}	3.7038	3.87481×10^{-3}	1.9875
320	1.74001×10^{-6}	4.3098	1.93917×10^{-3}	1.9982

Immersed Interface Method

- Represent interface by spline through smaller number of control points:

- Can compute force and hence pressure jump at any point on membrane.

$$f(s,t) = \frac{\partial}{\partial s}(T(s,t)\tau(s,t)),$$

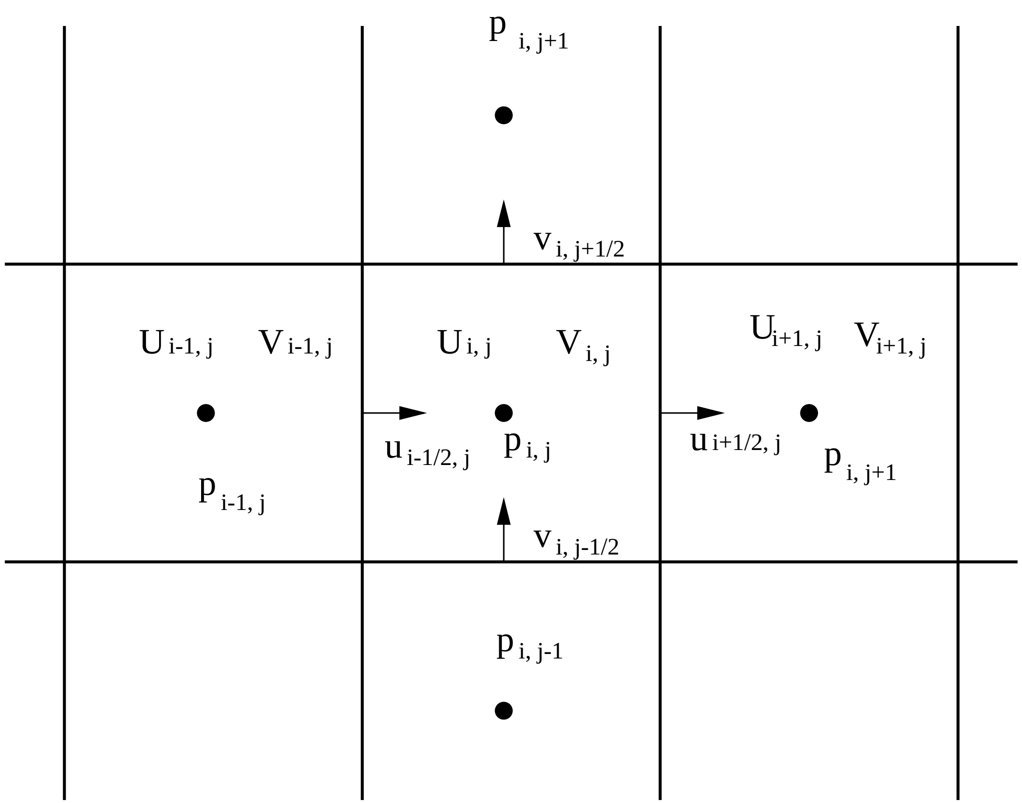
where

$$T(s,t) = T_0\left(\left|\frac{\partial X(s,t)}{\partial s}\right| - 1\right),$$

- Incorporate jump conditions into Taylor series expansion to derive finite-difference method that is pointwise second-order accurate.

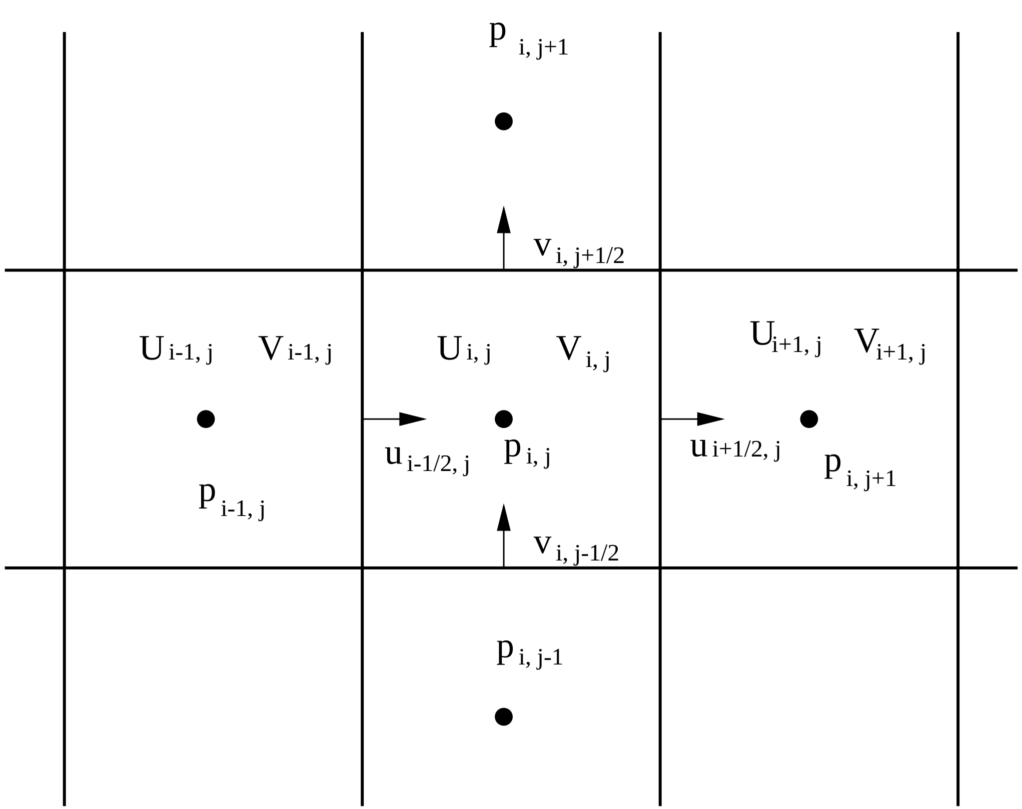
Hybrid Finite-Volume / Finite-Difference Method

- Fluid equations solved on finite-volume grid.
- Cell-centered velocity advected using high-resolution methods (CLAWPACK)
- Edge velocities needed in advection algorithm are obtained by averaging cell-centered values.



Hybrid Finite-Volume / Finite-Difference Method

- Divergence-free condition applied to the edge velocities.
- Finite-difference immersed interface method used to compute pressure at cell centers.
- Pressure correction is then applied to cell-centered velocities.



Explicit Method

Given X_k^n, U^n, u^n at start of time step.

Step 1. Solve $U_t + (u^n \cdot \nabla)U = 0$. Takes $U^n \; \rightarrow \; U^\dagger$.

Step 2. Solve $U_t = \mu \nabla^2 U + f_\tau$. Takes $U^\dagger \; \rightarrow \; U^*$.

Step 3. Average U^* from adjacent grid cells to obtain u^* at edges.

Step 4. Solve $\Delta t \, \nabla^2 p^{n+1} = \nabla \cdot u^*$ with $[\![p^{n+1}]\!] = f_n$ to obtain p^{n+1}.

Step 5. Update u^* based on p^{n+1} to get u^{n+1} (div free).

Step 6. Update U^* based on p^{n+1} to get U^{n+1}.

Step 7. Interpolate U^{n+1} to marker points X^n to obtain $\mathcal{U}^{n+1}(X^n)$. Move membrane using

$$X^{n+1} = X^n + \Delta t \, \mathcal{U}^{n+1}(X^n).$$

Trapezoidal Method

$\mathcal{U}^n(X^n)$ = velocities at marker points X^n, interpolated from U^n,

$\mathcal{U}^{n+1}(X^{n+1})$ = velocities at marker points X^{n+1}, interpolated from

U^{n+1}, determined by stepping forward from U^n with forces

$\frac{1}{2}(f(X^n) + f(X^{n+1}))$.

Would like:

$$X^{n+1} = X^n + \frac{1}{2}\Delta t\,(\mathcal{U}^n(X^n) + \mathcal{U}^{n+1}(X^{n+1})).$$

Problem: Implicit in X^{n+1} and computing $\mathcal{U}^{n+1}(X^{n+1}))$ requires solving fluid equations and Poisson problem.

Implicit Method

Given X_k^n, U^n, u^n at start of time step.
Apply Quasi-Newton method to solve for X^{n+1}.

Step I1. Apply Step 1, the advection step. $U^n \;\rightarrow\; U^\dagger$.

Step I2. Make a guess $X^{[0]}$ for X^{n+1} and set $I = 0$.

Step I3. Perform Steps 2–6 but replacing $f(X^n)$ by $\frac{1}{2}(f(X^n) + f(X^{[I]}))$
This gives provisional velocity field U^{n+1}.

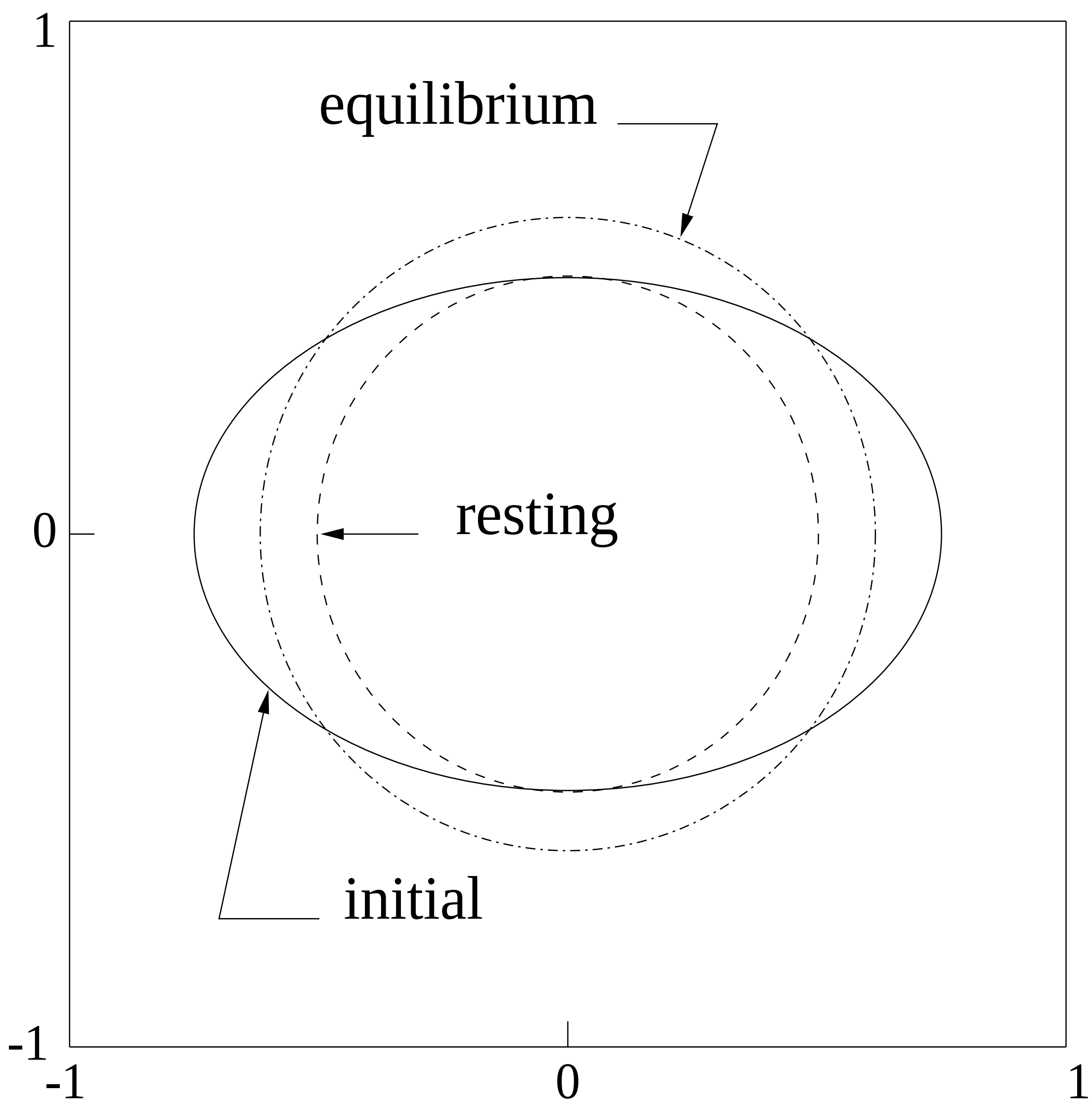
Step I4. Evaluate

$$g(X^{[I]}) = X^{[I]} - X^n - \frac{1}{2}\Delta t\,(\mathcal{U}^n(X^n) + \mathcal{U}^{n+1}(X^{[I]}))$$

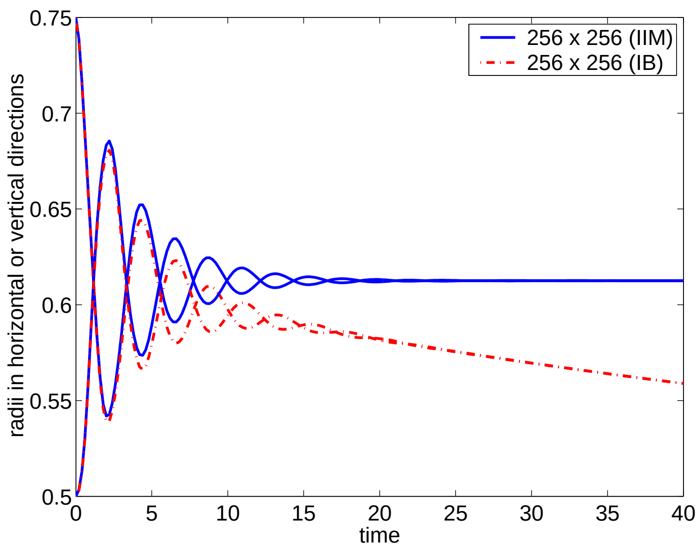
Step I5. Convergence check: If $\|g(X^{[I]})\| \leq \epsilon$ then set $X^{n+1} = X^{[I]}$, done.

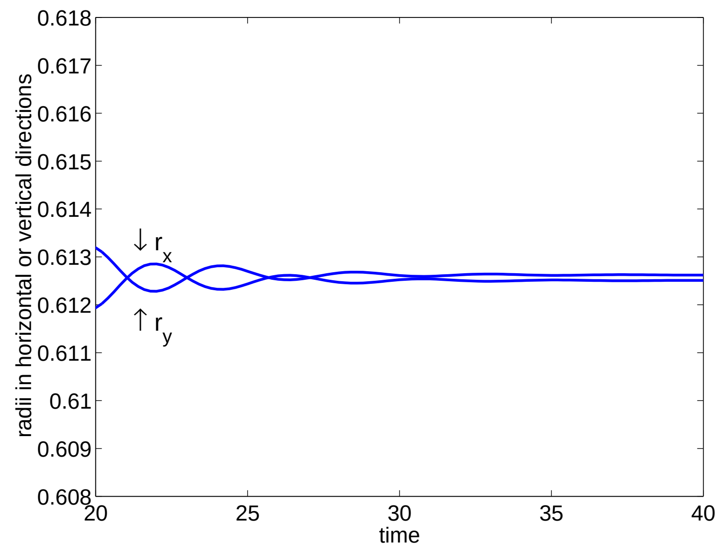
Otherwise, update $X^{[I]}$ to $X^{[I+1]}$, set $I = I + 1$, and go to step I3.

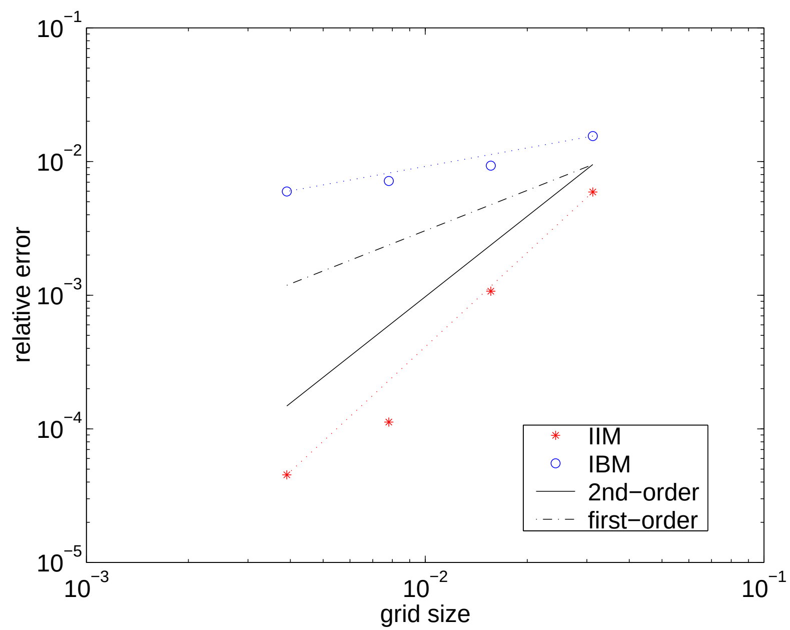
Oscillating balloon



Volume Conservation and Accuracy

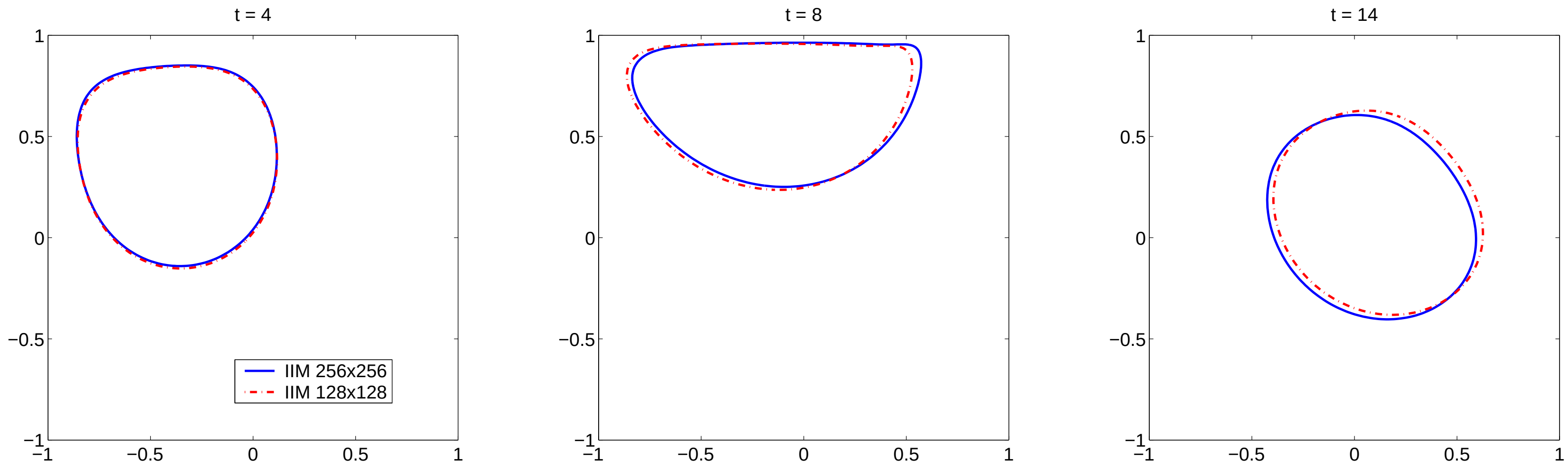




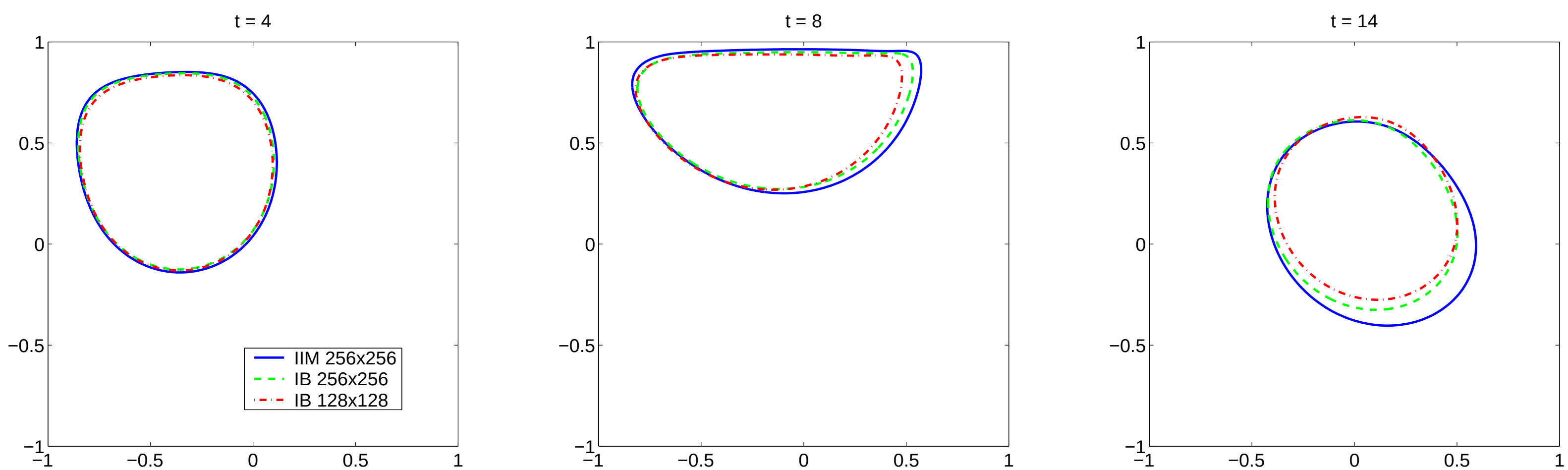


Balloon in driven cavity

IIM:



IBM:



Summary

- High-resolution finite volume for convective terms
- Finite difference immersed interface method for pressure Poisson problem
- Tangential component of force is spread using discrete delta functions
- Normal component of force gives jump in pressure, built into Poisson solver
- Easy modification of existing immersed boundary codes?
- Implicit method for moving the boundary
- Membrane mass can also be incorporated

Adding mass to the membrane

$$m(s)X_{tt}(s,t) = f(s,t) - [\![p]\!]\vec{n} + \mu\left[\frac{\partial u}{\partial n}\right].$$

Force now used in pressure and fluid solve:

$$\tilde{f}(s,t) = f(s,t) - m(s)X_{tt}(s,t).$$

Split this into normal and tangental components.

Easy to incorporate into implicit algorithm:

Step I3. Replace $\frac{1}{2}(f(X^n) + f(X^{[I]}))$ by

$$\frac{1}{2}(f(X^n) + f(X^{[I]})) - \frac{m(s)}{\Delta t^2}(X^{[I]} - 2X^n + X^{n-1}).$$