Solitary Waves in Layered Nonlinear Media

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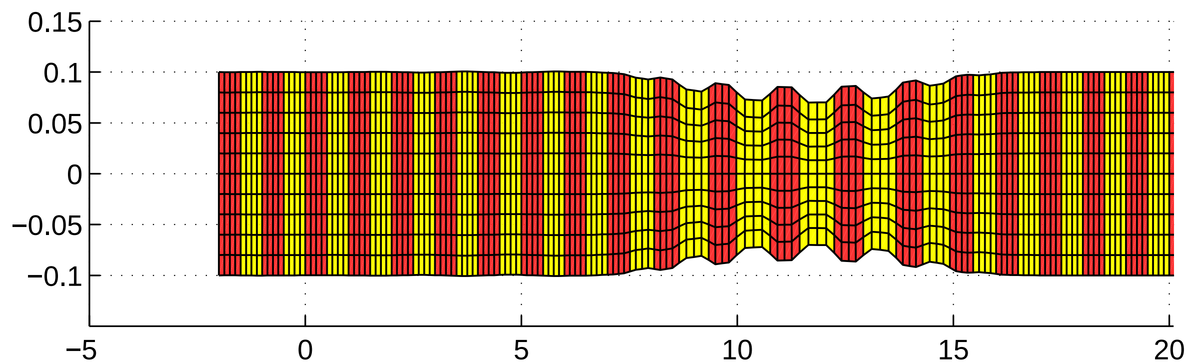


One-dimensional Elasticity

Notation:

$$X(x,t) = \text{location of particle indexed by } x \text{ in the reference (unstrained) configuration}$$

$$X(x,0) = x \text{ if initially unstrained}$$

$$\epsilon(x,t) = X_x(x,t) - 1 = \text{ strain}$$

$$u(x,t) = \text{velocity of particle indexed by } x$$

$$\rho(x) = \text{density}$$

Constitutive relation

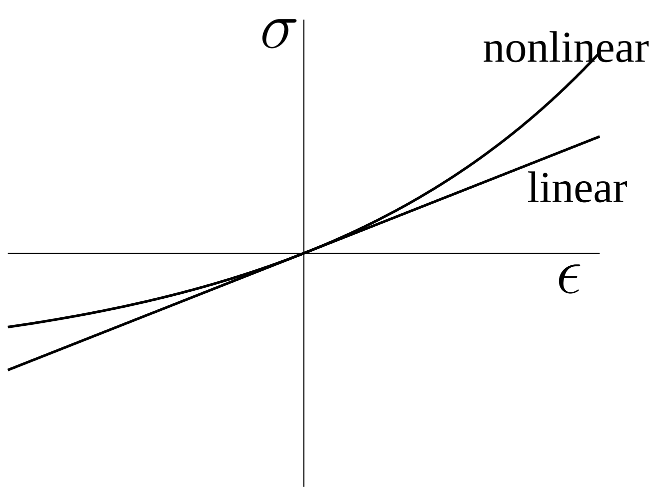
stress $= \sigma(\epsilon, x)$, where $\epsilon =$ strain

Heterogeneous $\implies$ explicit dependence on x.

Linear elasticity: (Hooke's law)

$$\sigma(\epsilon, x) = K(x)\epsilon$$

where K is the bulk modulus of compressibility.



System of two conservation laws:

$$
\begin{aligned}
\epsilon_t - u_x &= 0 \\
(\rho u)_t + \sigma_x &= 0
\end{aligned}
$$

or $q_t + f(q,x)_x = 0$ with

$$
q = \begin{bmatrix} \epsilon \\ \rho u \end{bmatrix} = \begin{bmatrix} q^1 \\ q^2 \end{bmatrix}, \qquad f(q,x) = \begin{bmatrix} q^2/\rho(x) \\ -\sigma(q^1,x) \end{bmatrix}
$$

Quasi-linear hyperbolic form:

$$
q_t + f_q(q,x)q_x = -f_x(q,x)
$$

Jacobian: $f_q(q,x) = \begin{bmatrix} 0 & -1/\rho(x) \\ -\sigma_\epsilon(\epsilon,x) & 0 \end{bmatrix}$

Eigenvalues (wave speeds): $\pm\sqrt{\sigma_\epsilon/\rho}$.

Wave-propagation algorithms for spatially-varying fluxes

$$q_t + f(q, x)_x = 0$$

Work with Derek Bale, Sorin Mitran, and James Rossmanith,

> *A wave-propagation method for conservation laws and balance laws with spatially varying flux functions*, SISC 24 (2002), pp. 955-978

Connections to relaxation schemes (Jin and Xin):
Work with Marica Pelanti,

> *A Class of Approximate Riemann Solvers and Their Relation to Relaxation Schemes*, J. Comput. Phys., 172 (2001), pp. 572-591.

Applications:

- Wave propagation in heterogeneous nonlinear media
- Flow in heterogeneous porous media
- Traffic flow with varying road conditions
- Solving conservation laws on curved manifolds

CLAWPACK

`http://www.amath.washington.edu/~claw/`

- Fortran codes with Matlab graphics routines.
- Many examples and applications to run or modify.
- 1d, 2d, and 3d.

User supplies:

- Riemann solver, splitting data into waves and fluctuations
  (Need not be in conservation form)
- Boundary condition routine to extend data to ghost cells
  Standard `bc1.f` routine includes many standard BC's
- Initial conditions — `qinit.f`

Linear case: $\sigma = K\epsilon$

$q_t + Aq_x = 0$ with $q = \begin{bmatrix} \epsilon \\ m \end{bmatrix}$, $A = \begin{bmatrix} 0 & -1/\rho \\ -K & 0 \end{bmatrix}$.

Diagonalize A: $R^{-1}AR = \Lambda$ where $\Lambda = \begin{bmatrix} -c & 0 \\ 0 & c \end{bmatrix}$, $c = \sqrt{K/\rho}$,

$$R = \begin{bmatrix} 1 & 1 \\ Z & -Z \end{bmatrix}, \qquad R^{-1} = \frac{1}{2Z}\begin{bmatrix} Z & 1 \\ Z & -1 \end{bmatrix}$$

$Z =$ impedance $= \rho c = \sqrt{K\rho}$.

System can be written as

$$R^{-1}q_t + R^{-1}AR\,R^{-1}q_x = 0 \qquad \text{or} \qquad w_t + \Lambda w_x = 0.$$

Linear case

With $w = R^{-1}q$,

$$w_t + \Lambda w_x = 0.$$

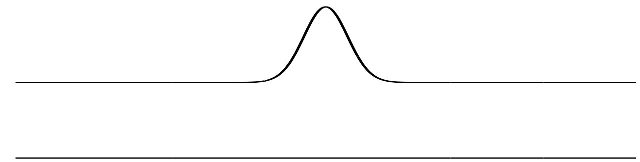
This is a decoupled pair of advection equations for the characteristic variables:

$$\begin{aligned} w_t^1 - cw_x^1 = 0 \quad &\Longrightarrow \quad w^1(x,t) = w^1(x+ct,0) \\ w_t^2 + cw_x^2 = 0 \quad &\Longrightarrow \quad w^2(x,t) = w^2(x-ct,0) \end{aligned}$$

$q(x,t)$ is a linear comb. of left-going and right-going waves:

$$q(x,t) = Rw(x,t) = w^1(x,t)\begin{bmatrix} 1 \\ Z \end{bmatrix} + w^2(x,t)\begin{bmatrix} 1 \\ -Z \end{bmatrix}.$$

Example: $q(x,0) = \begin{bmatrix} H(x) \\ 0 \end{bmatrix}$



Then

$$\begin{aligned}
w(x,0) &= R^{-1} q(x,0) = \frac{1}{2} \begin{bmatrix} H(x) \\ H(x) \end{bmatrix} \\
q(x,t) &= R w(x,t) = w^1(x,t) \begin{bmatrix} 1 \\ Z \end{bmatrix} + w^2(x,t) \begin{bmatrix} 1 \\ -Z \end{bmatrix} \\
&= \frac{1}{2} H(x+ct) \begin{bmatrix} 1 \\ Z \end{bmatrix} + \frac{1}{2} H(x-ct) \begin{bmatrix} 1 \\ -Z \end{bmatrix}
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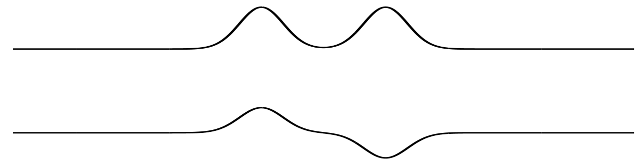

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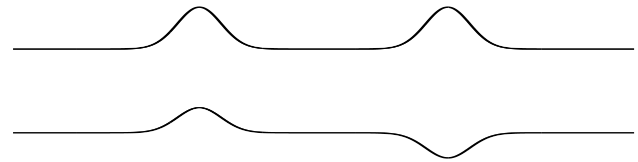

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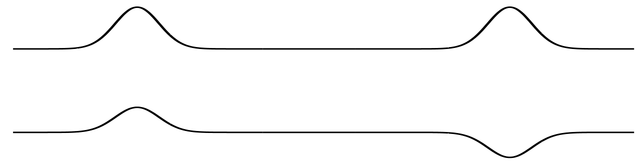

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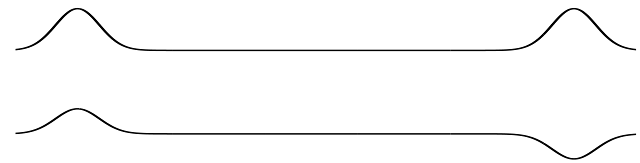

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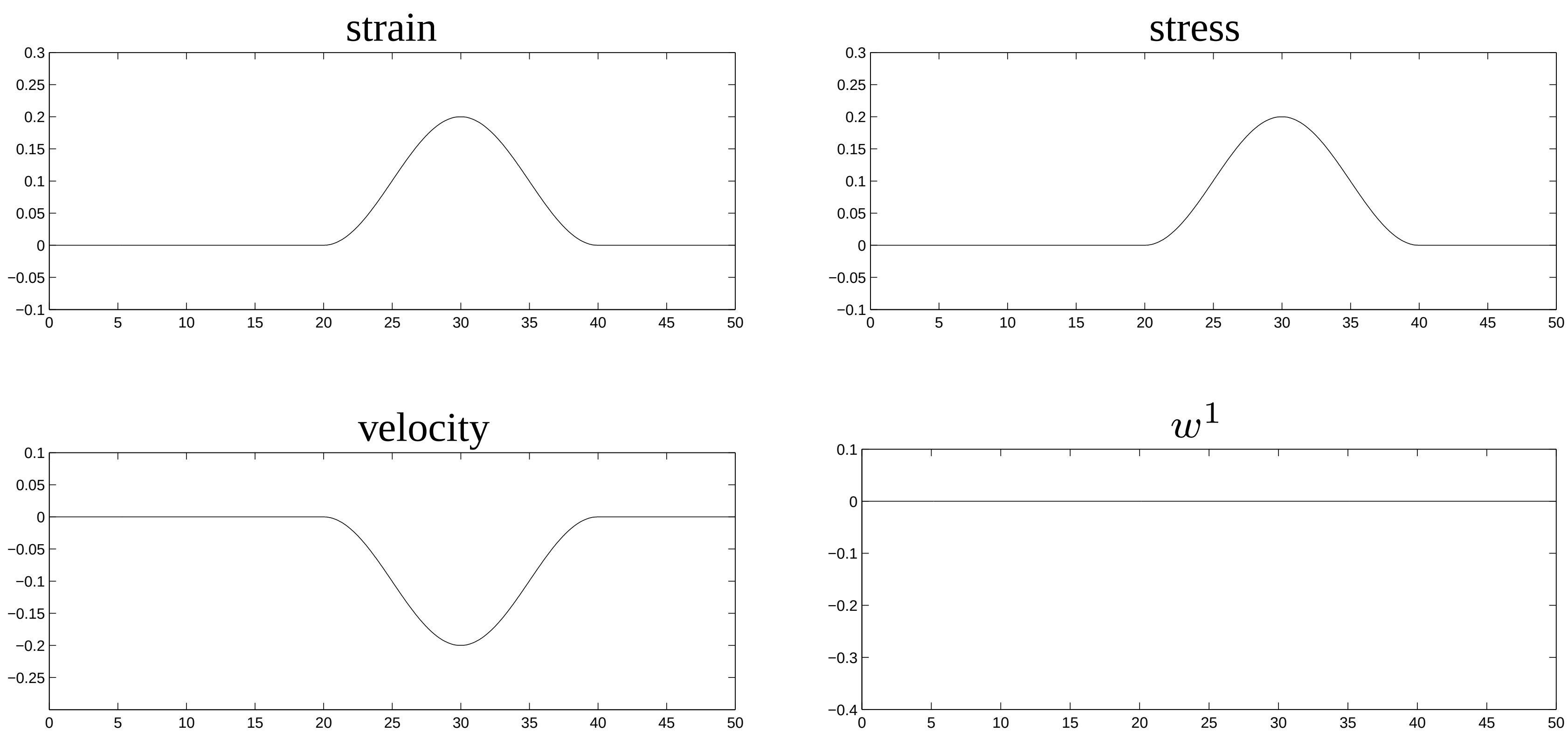
$$

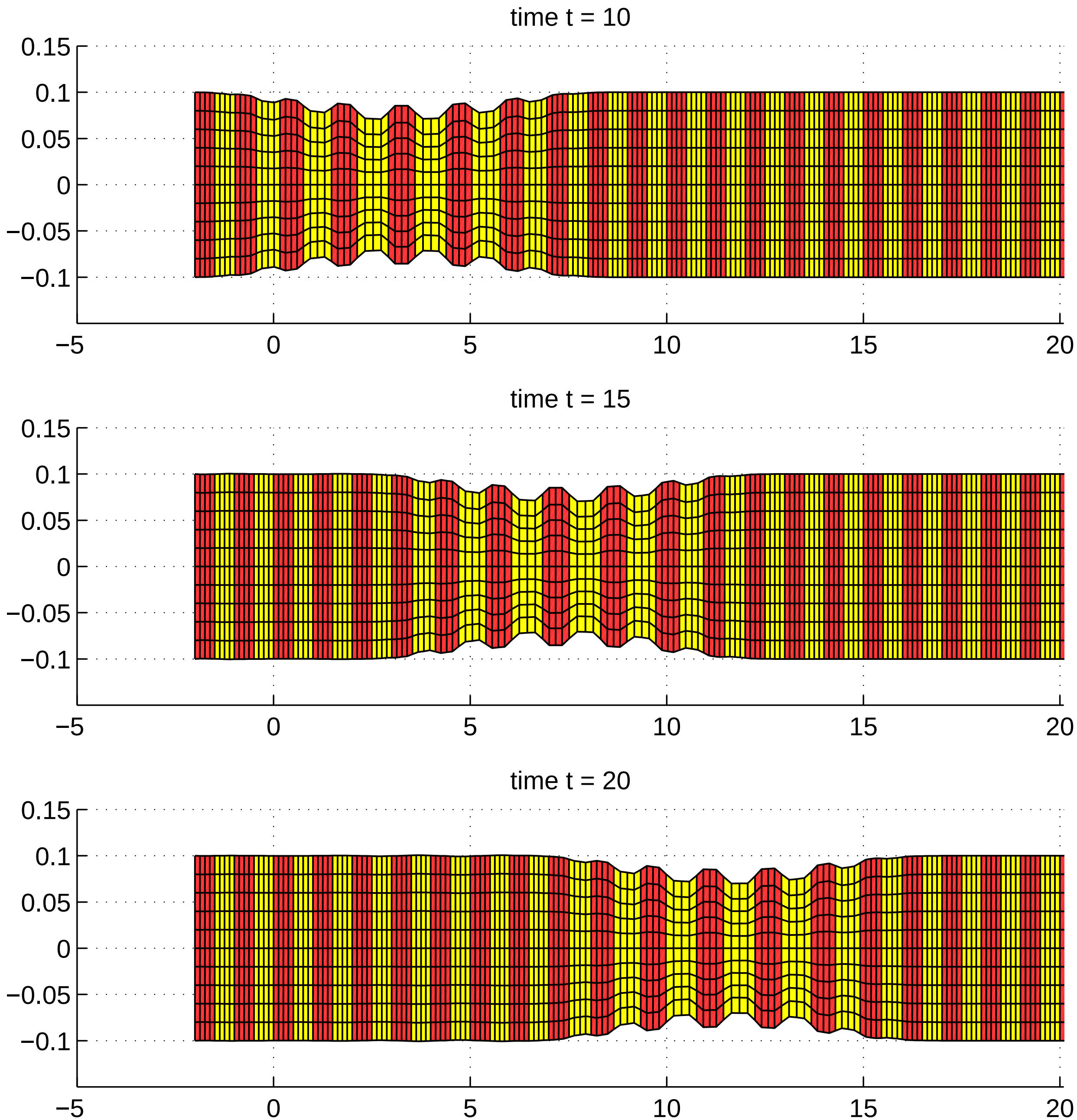
Waves in a homogeneous linear medium



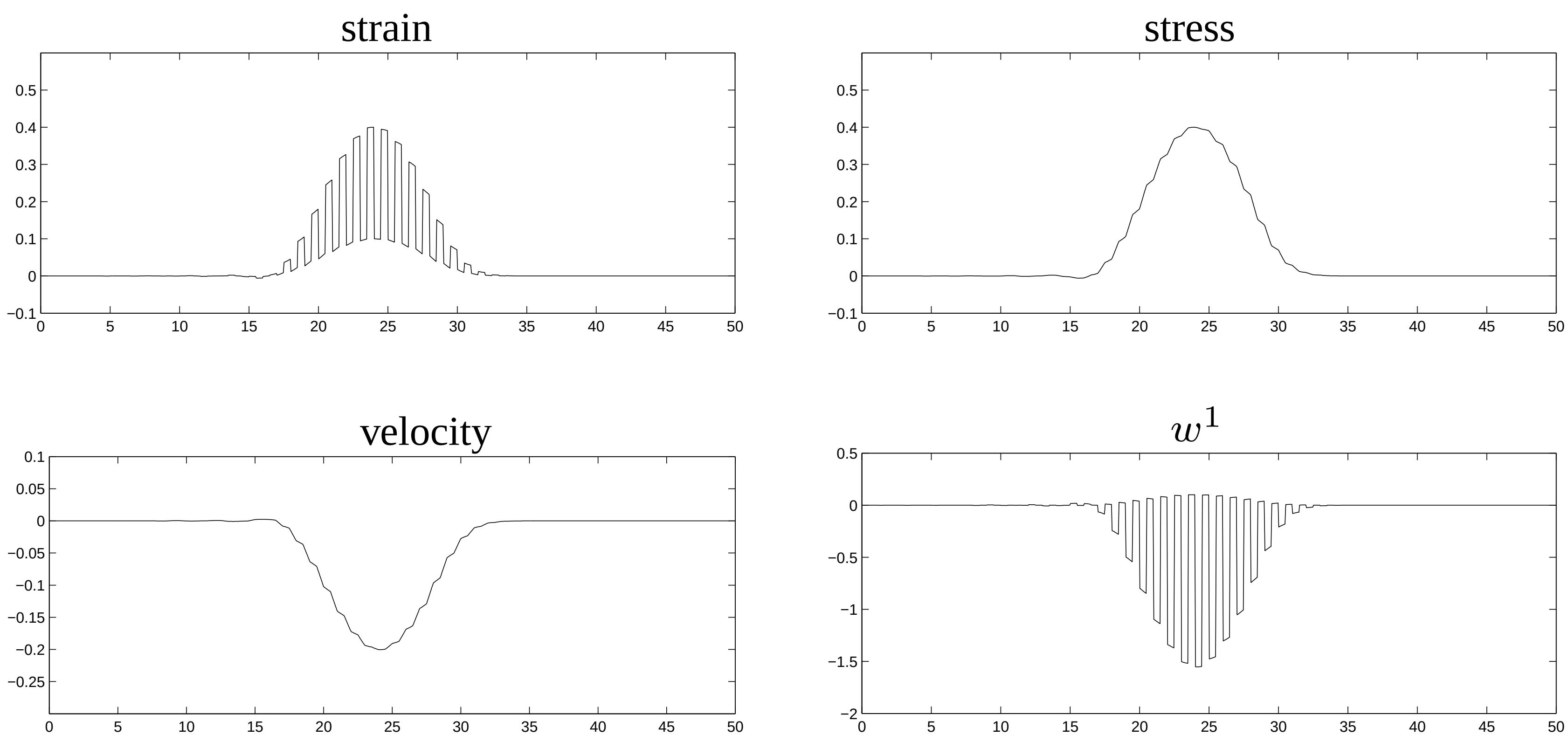
$c \equiv 1, \quad Z \equiv 1$

At $t = 40$ the leading edge of wave is at $x = 40$.

Waves in a layered elastic plate



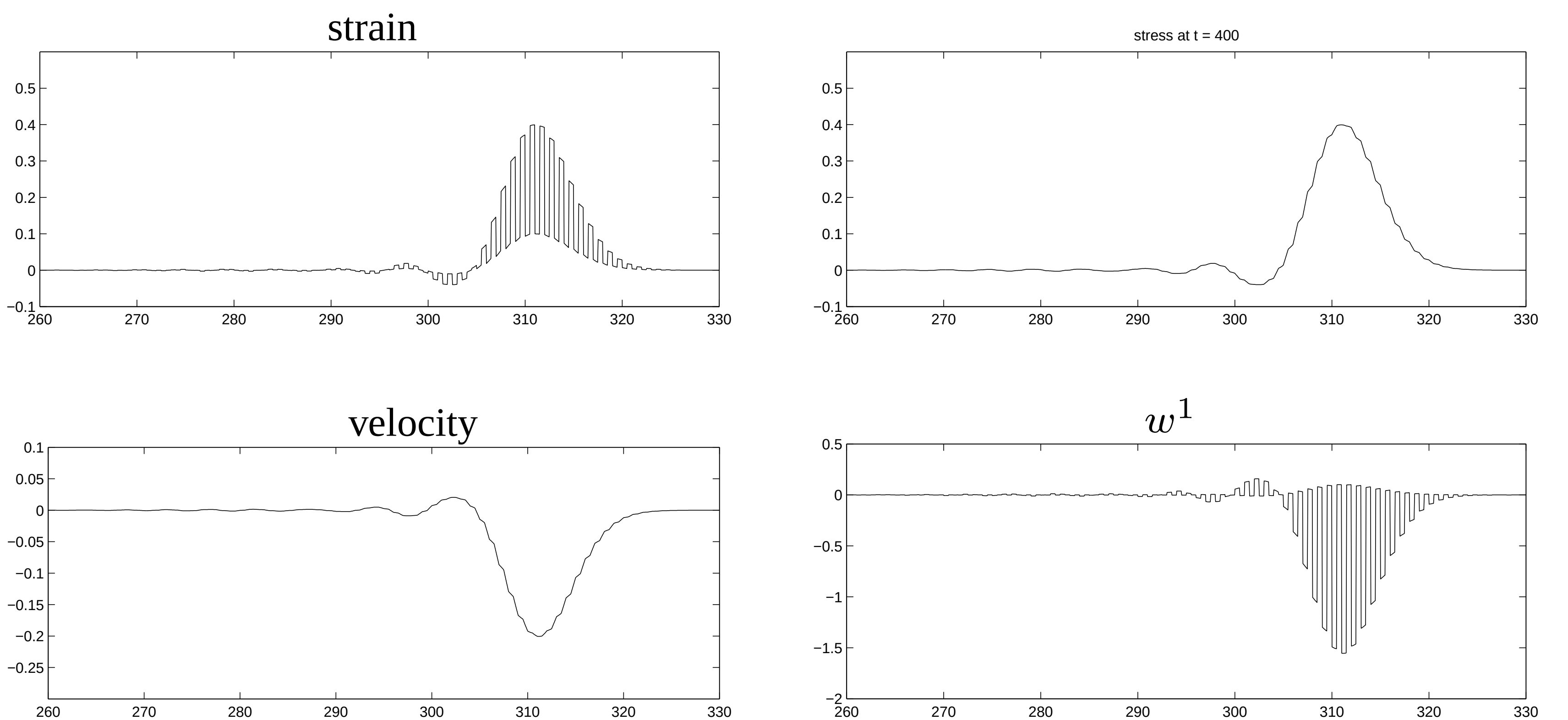
Waves in a heterogeneous linear medium



$c \equiv 1$ but Z is different in the layers.

At $t = 40$ the leading edge of wave is at $x \approx 32$.

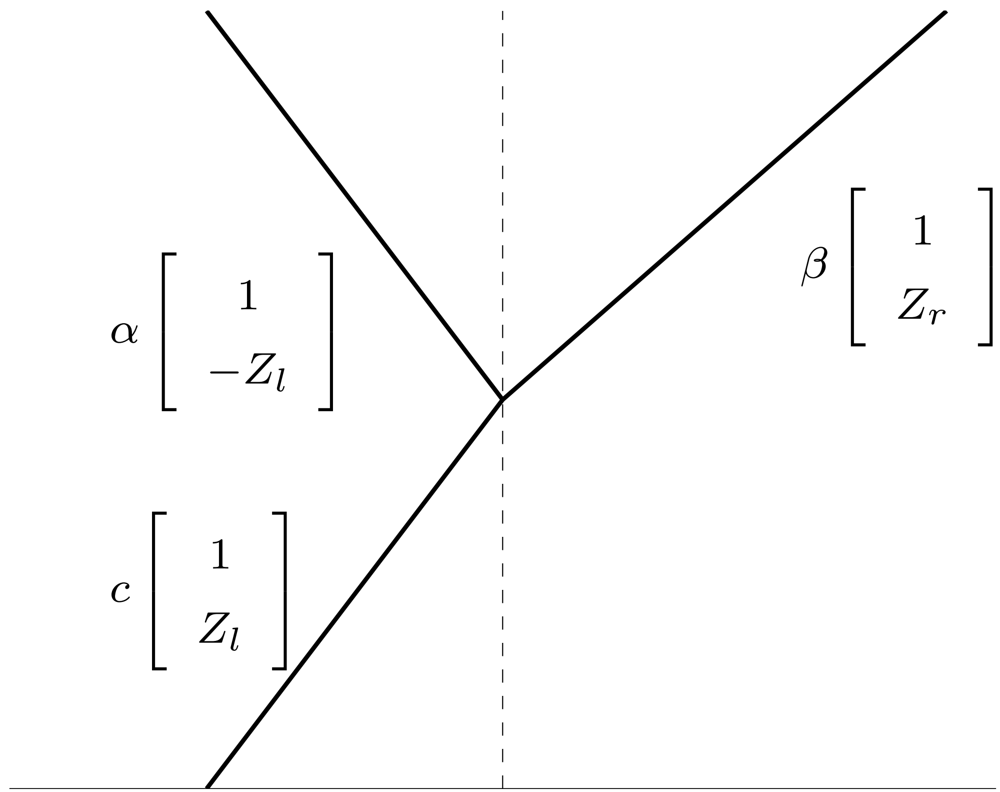
Waves in a heterogeneous linear medium, $t = 400$



Waves in layered medium

Note: Form of wave depends on impedance Z.

If Z varies with x then we expect reflection at interfaces



Waves in layered medium

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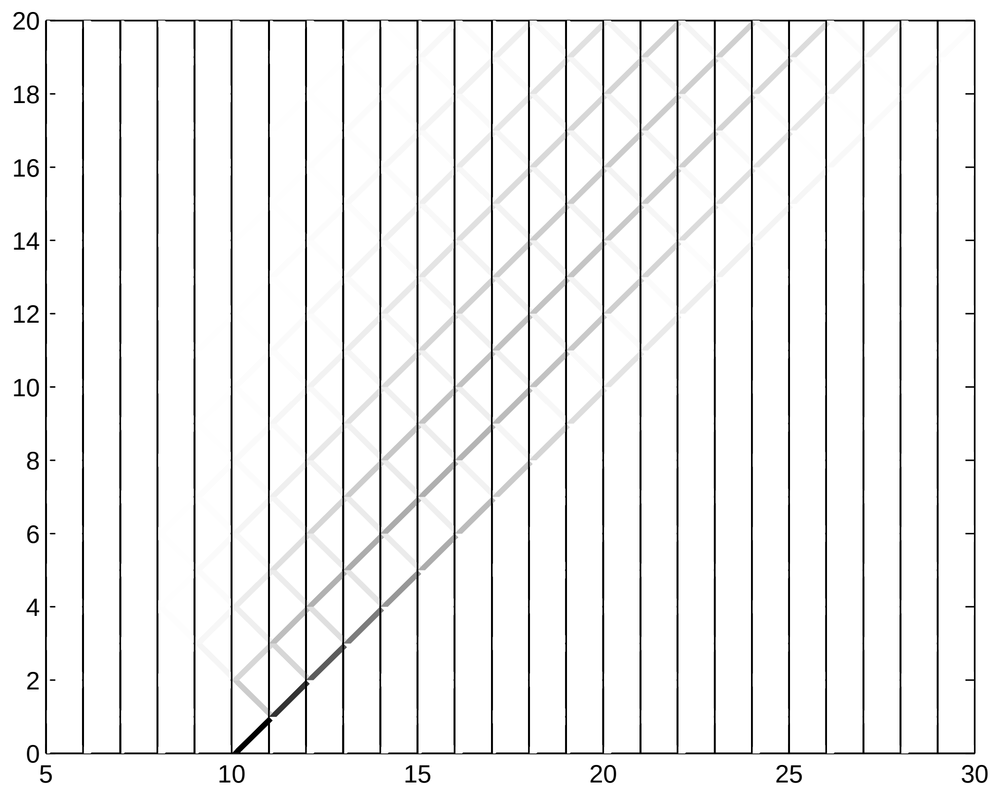

Nonlinear materials

$$\begin{aligned} \epsilon_t - u_x &= 0 \\ (\rho u)_t + \sigma_x &= 0 \end{aligned}$$

where

$$\sigma(\epsilon, x) = K(x)\epsilon + \beta K^2(x)\epsilon^2$$

or

$$\sigma(\epsilon, x) = e^{K(x)\epsilon} - 1 \approx K(x)\epsilon + \frac{1}{2}K^2(x)\epsilon^2$$

Layered medium can be related to Toda lattice

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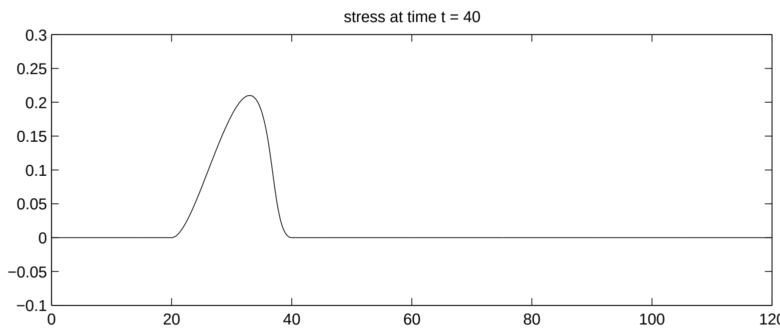
Layered medium can be related to Toda lattice

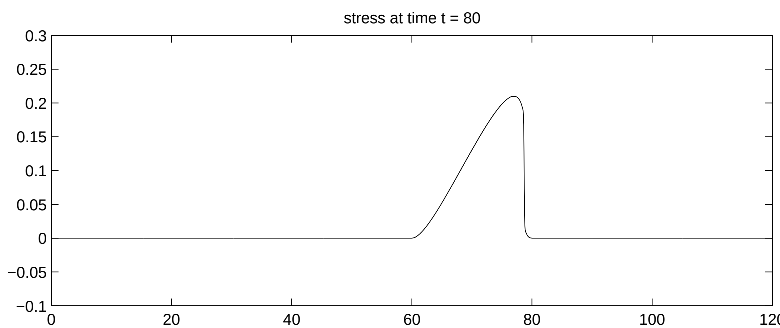
The wave speed increases with ϵ:

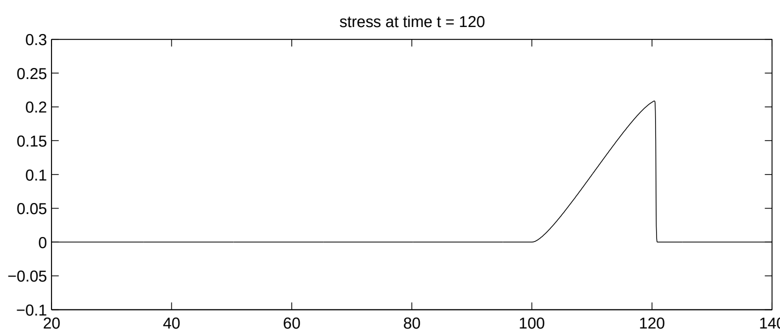
$$c(\epsilon, x) = \sqrt{\frac{\sigma_\epsilon(\epsilon, x)}{\rho(x)}} \approx \sqrt{\frac{K(x)(1+\epsilon)}{\rho(x)}}$$

Leads to shock formation in a homogeneous medium.

Nonlinear homogeneous medium





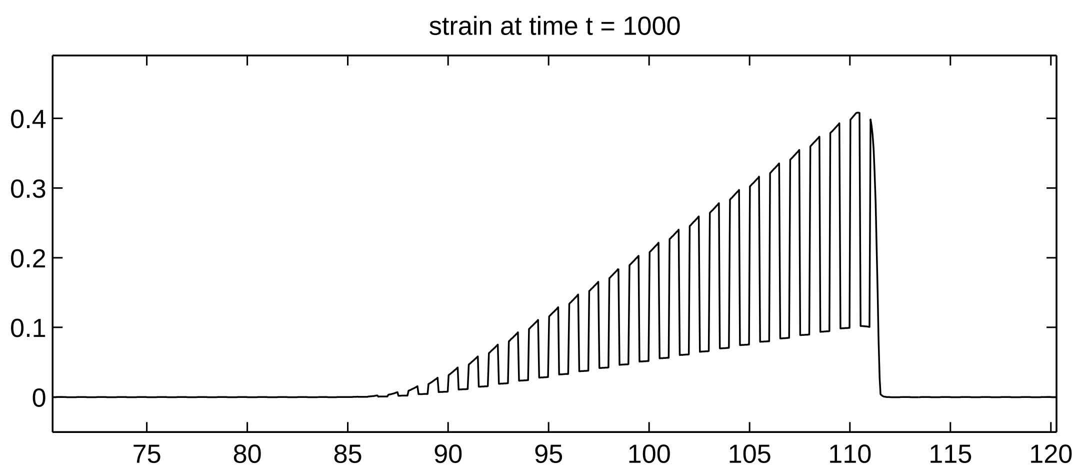


Nonlinear with constant linearized impedance

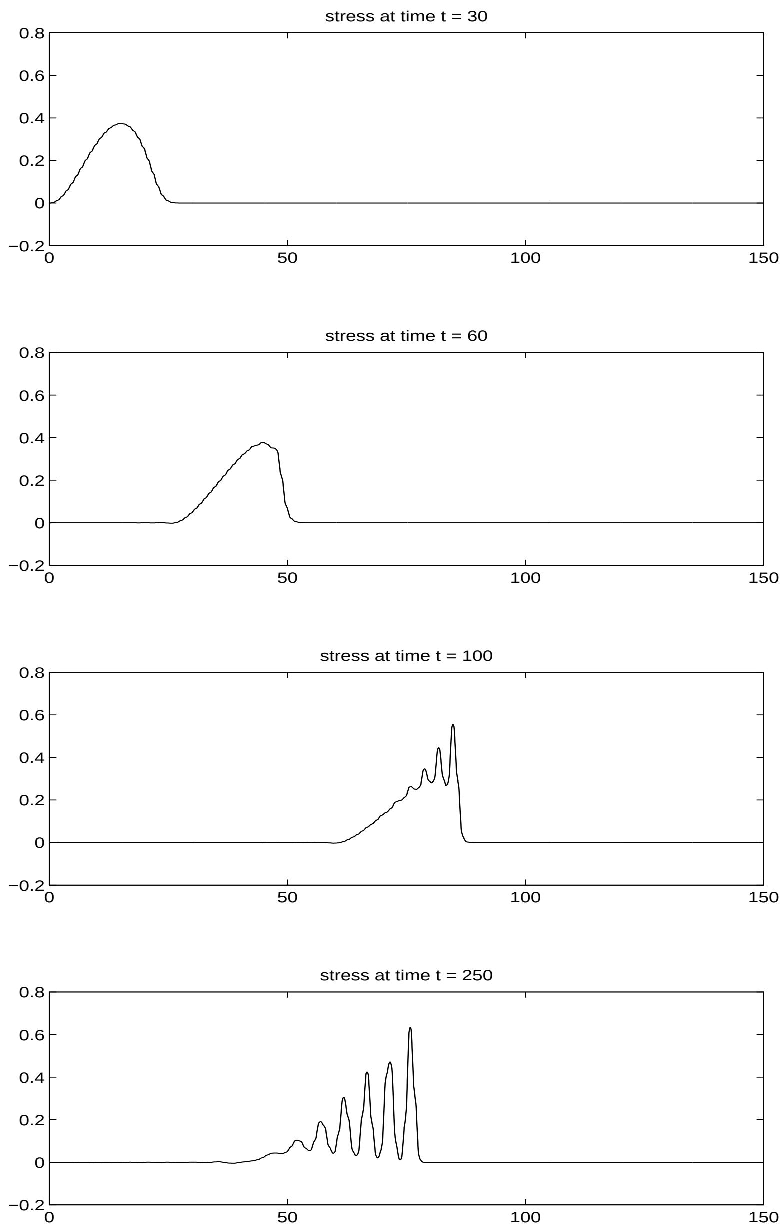
$$Z(x) = \rho(x)c(x) = \sqrt{\rho(x)K(x)(1+\epsilon)}$$

$$\longrightarrow \quad \sqrt{\rho(x)K(x)} \text{ as } \epsilon \to 0$$

Layered medium with $\rho(x)K(x) \equiv$ constant:



Layered nonlinear medium with $Z_A \neq Z_B$



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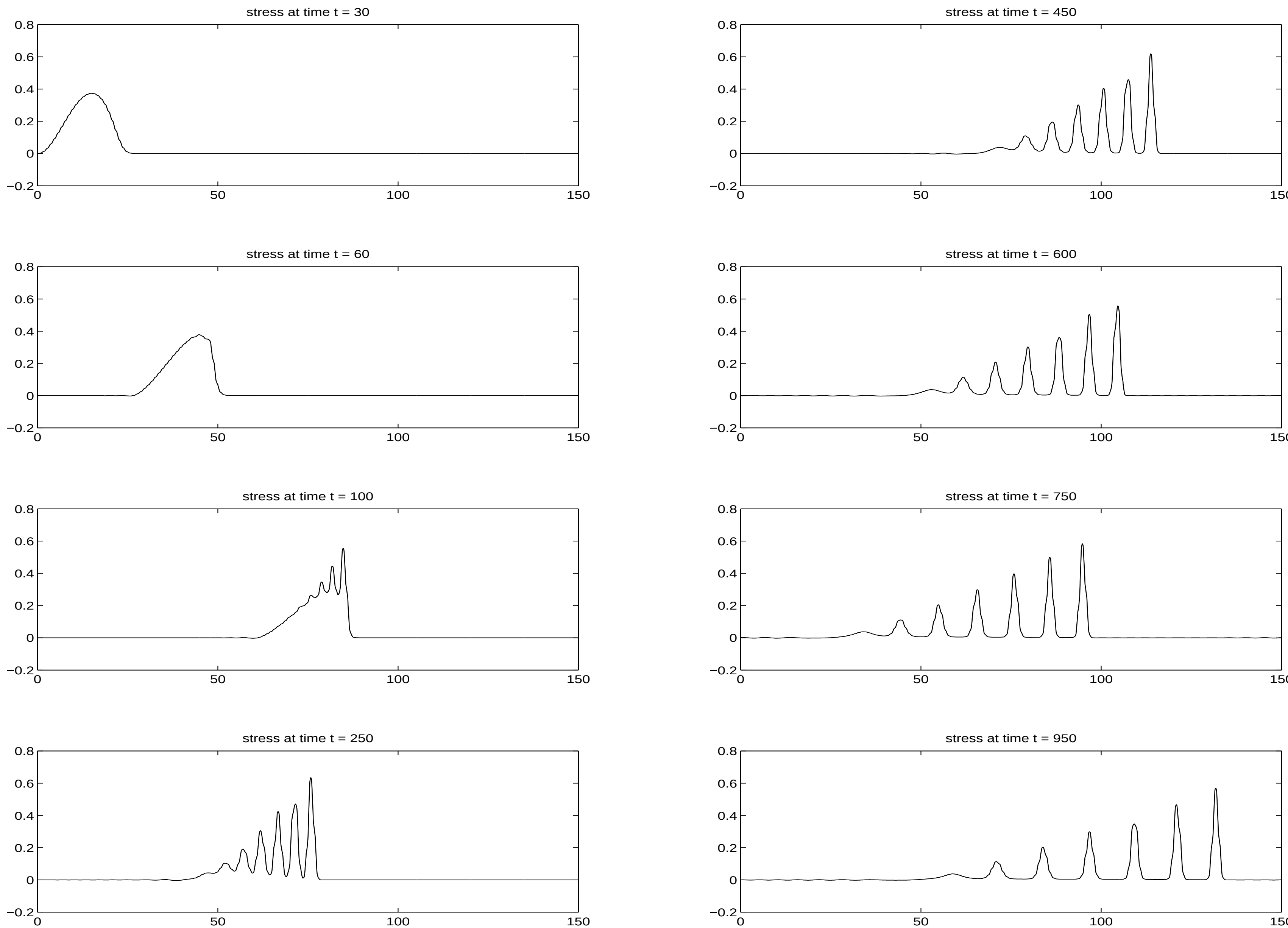
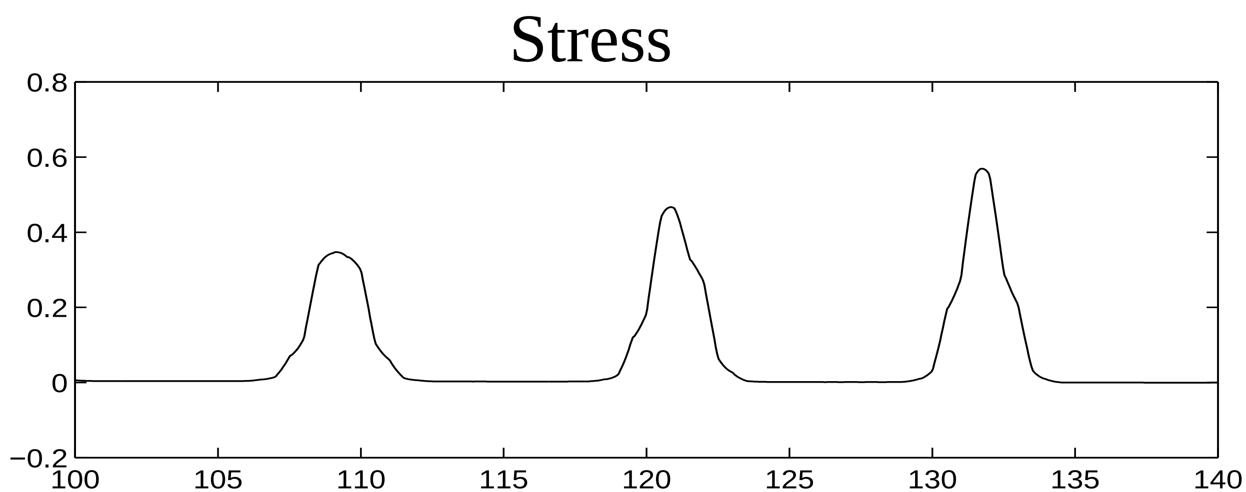
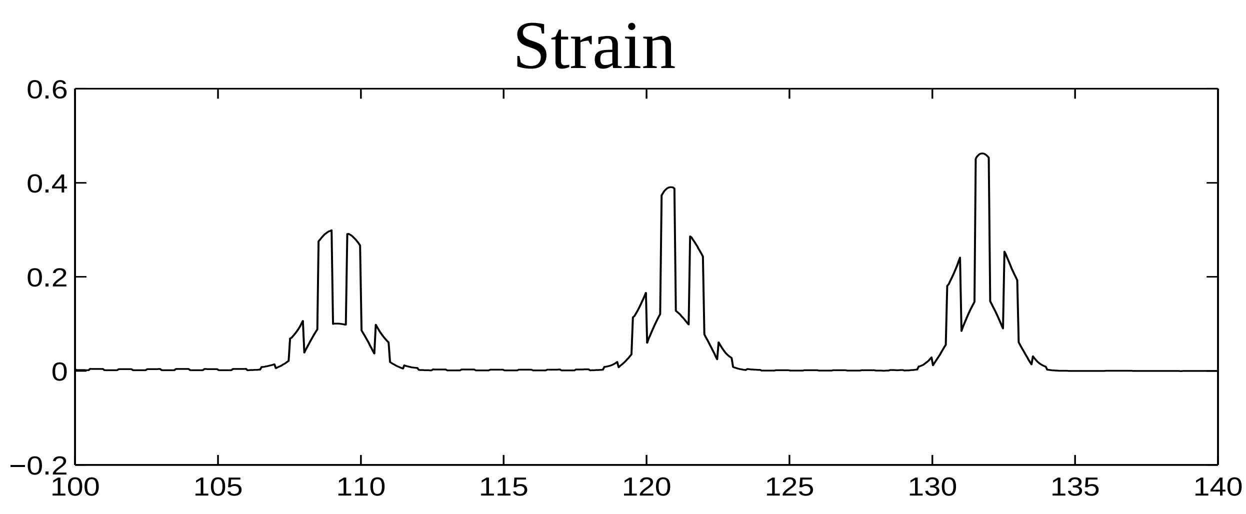
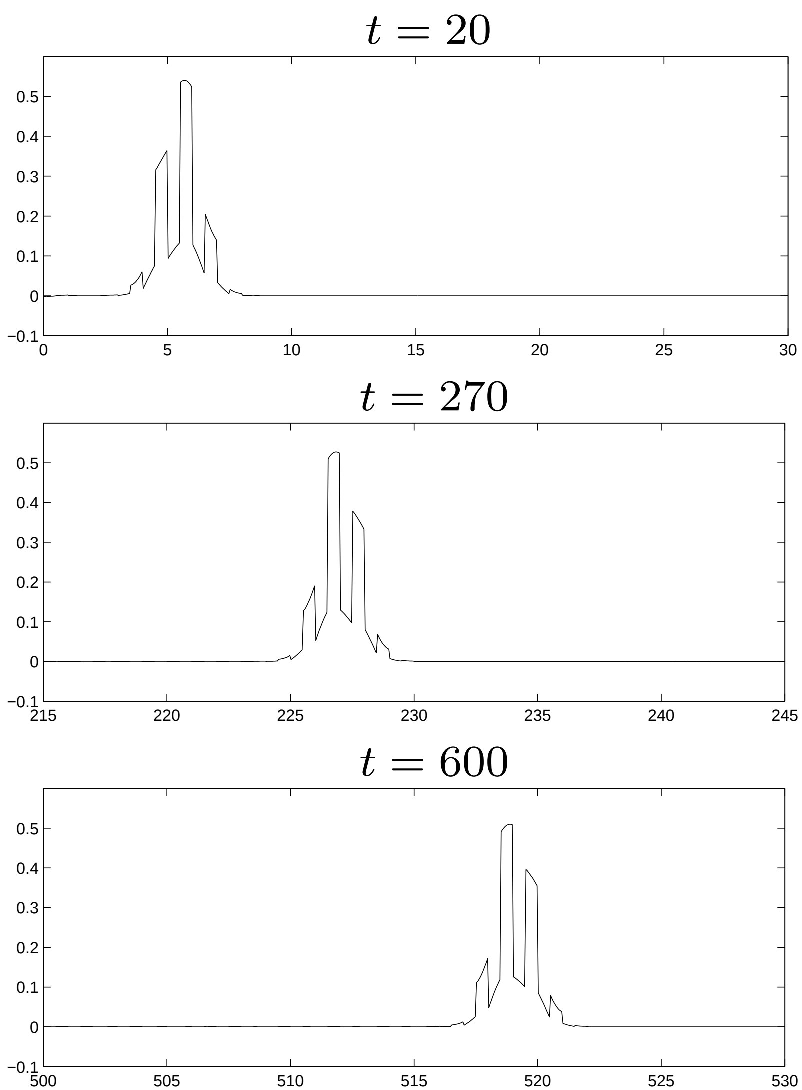


Zoomed view of 3 “stegotons”

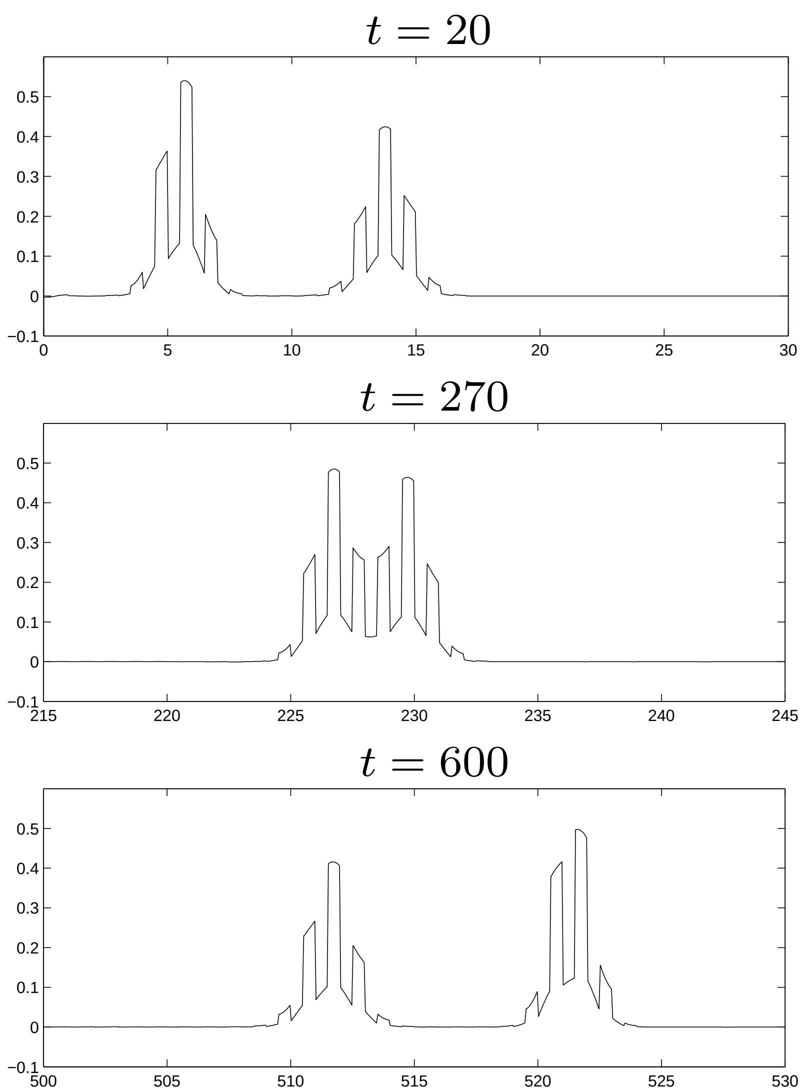




Propagation of a stegoton



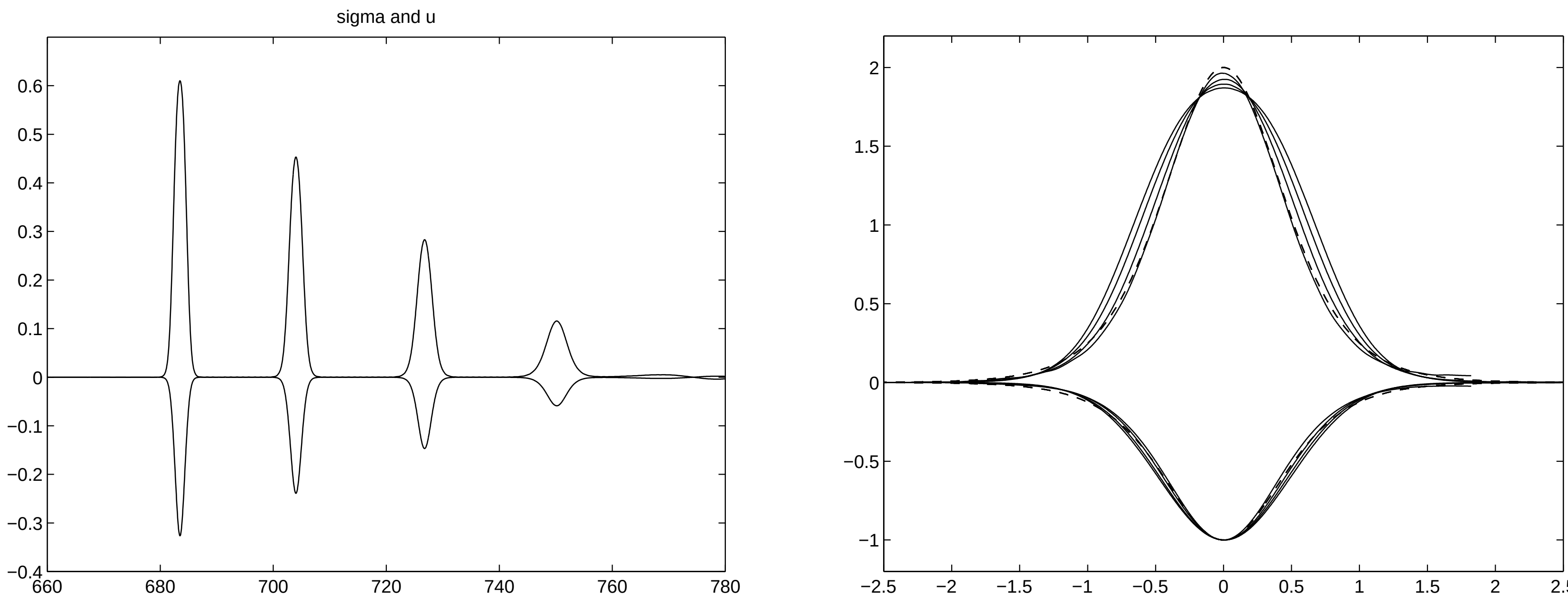
Collision of two stegotons



Collision of two stegotons

Movie:

Solution at fixed point x_0



On the right we plot

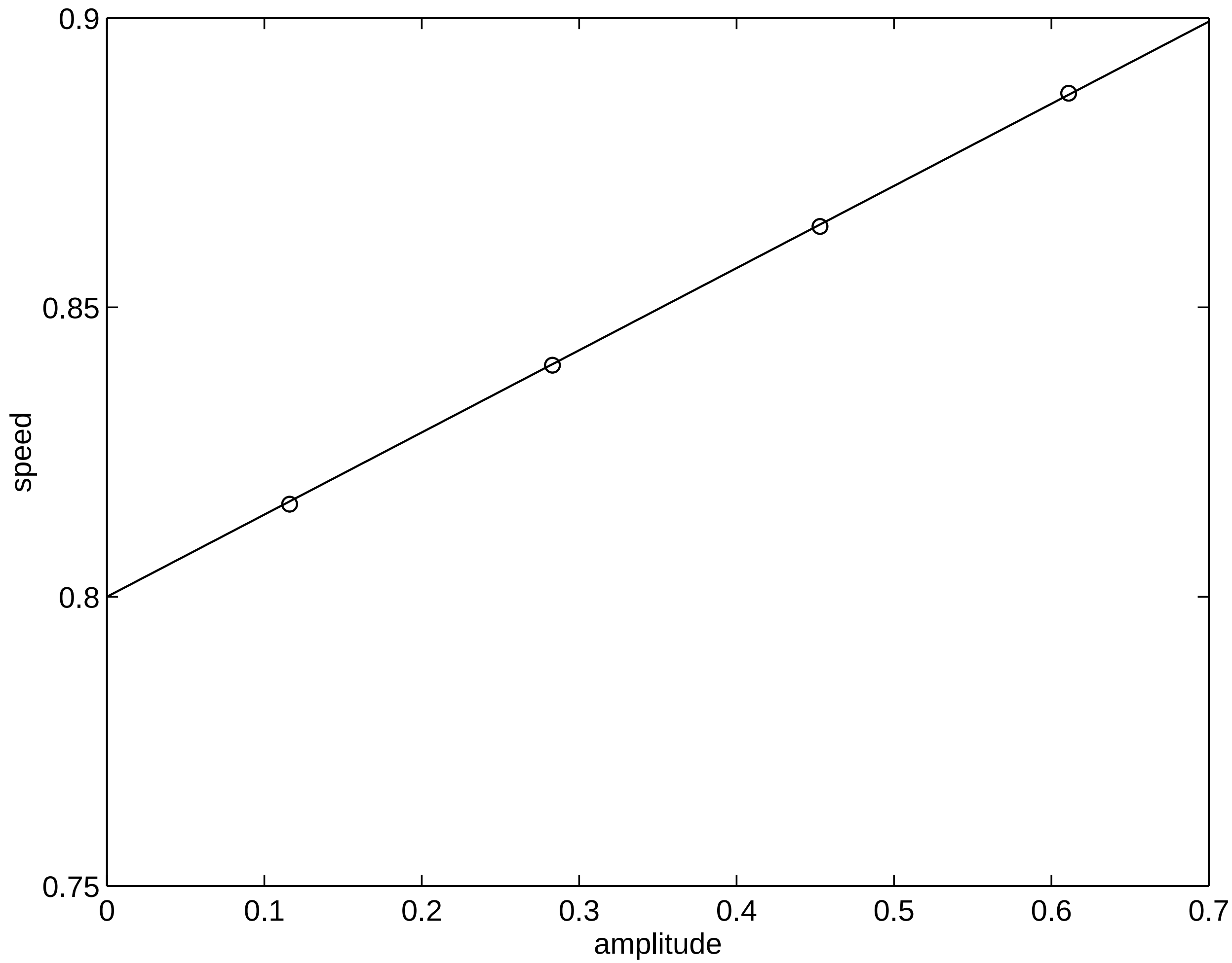
$$\frac{1}{a}\sigma(x_0,\tau) \qquad \text{and} \qquad \frac{1}{a}u(x_0,\tau)$$

as functions of

$$\tau = \sqrt{a}(t - t_m)$$

and t_m is the time the velocity reaches its peak value $-a$.

Plot of speed vs. amplitude



$$v = 0.8 + 0.142a.$$

Soliton behavior?

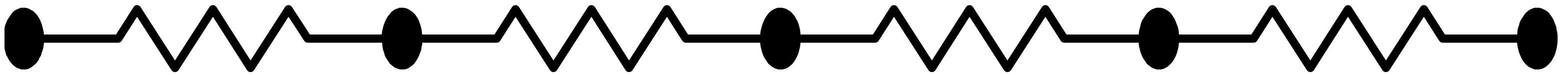
- Relation of layered medium to Toda lattice.

  Discrete nonlinear mass-spring system with soliton solutions.

- Nonlinear homogenization theory.

  Reduction to higher order nonlinear PDE with constant coefficients.

Lattice model for vibration of solids



Discrete particles coupled with springs:

$X_k(t)$ = location of kth particle, with mass m_k

$X_k(0) = k\Delta x$ undisturbed configuration ($\Delta x = 1$)

Strain of spring between particle k and $k+1$:

$$\begin{aligned}\epsilon_{k+1/2}(t) &= \frac{X_{k+1}(t) - X_k(t)}{\Delta x} - 1 \\ &= 0 \text{ if unstretched}\end{aligned}$$

Restoring force (= stress): $\sigma(\epsilon_{k+1/2})$

or $\sigma_{k+1/2}(\epsilon_{k+1/2})$ if heterogeneous

Velocity: $U_k(t) = X_k'(t)$

Lattice model for vibration of solids

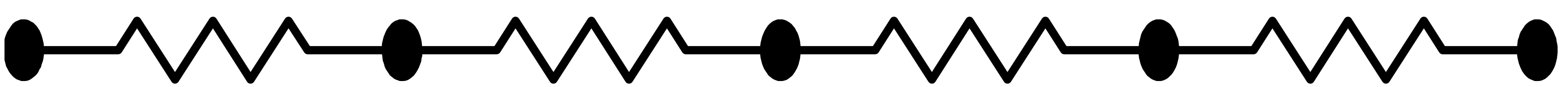
With $m_k = \Delta x \, \rho_k$, the equations of motion are

$$\begin{aligned} \epsilon'_{k+1/2}(t) &= \frac{U_{k+1}(t) - U_k(t)}{\Delta x} \\ \rho_k U'_k(t) &= \frac{\sigma(\epsilon_{k+1/2}(t)) - \sigma(\epsilon_{k-1/2}(t))}{\Delta x} \end{aligned}$$

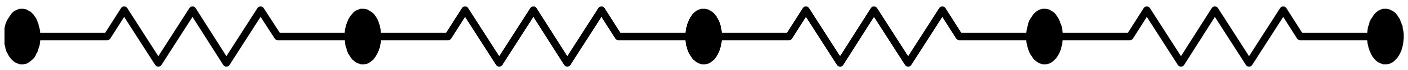
These can be viewed as a discretization of the continuum equations

$$\begin{aligned} \epsilon_t &= u_x \\ \rho u_t &= \sigma_x \end{aligned}$$

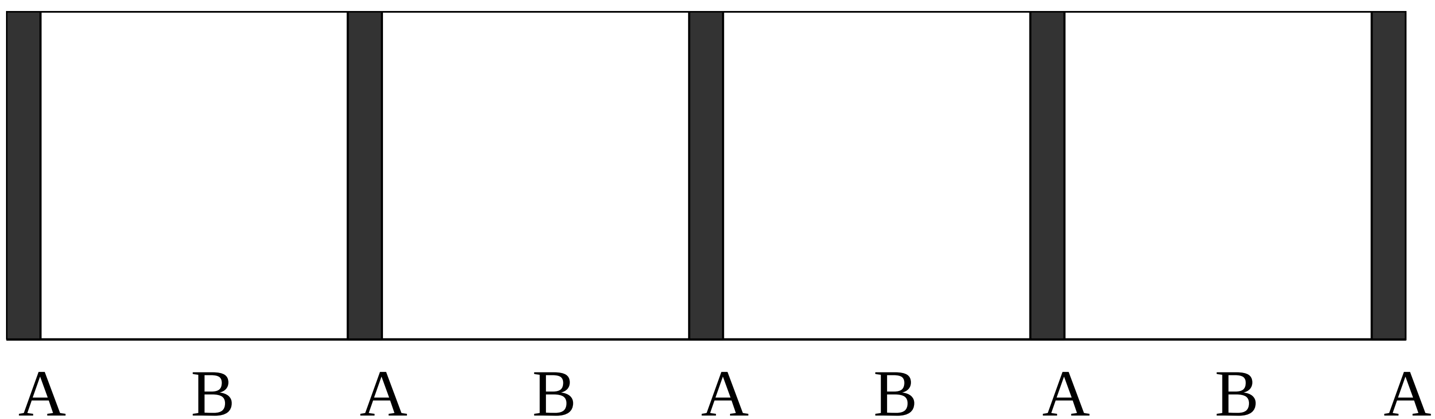
Linear springs: Hooke's law, $\sigma(\epsilon) = K\epsilon$.



Toda Lattice



Layered Medium



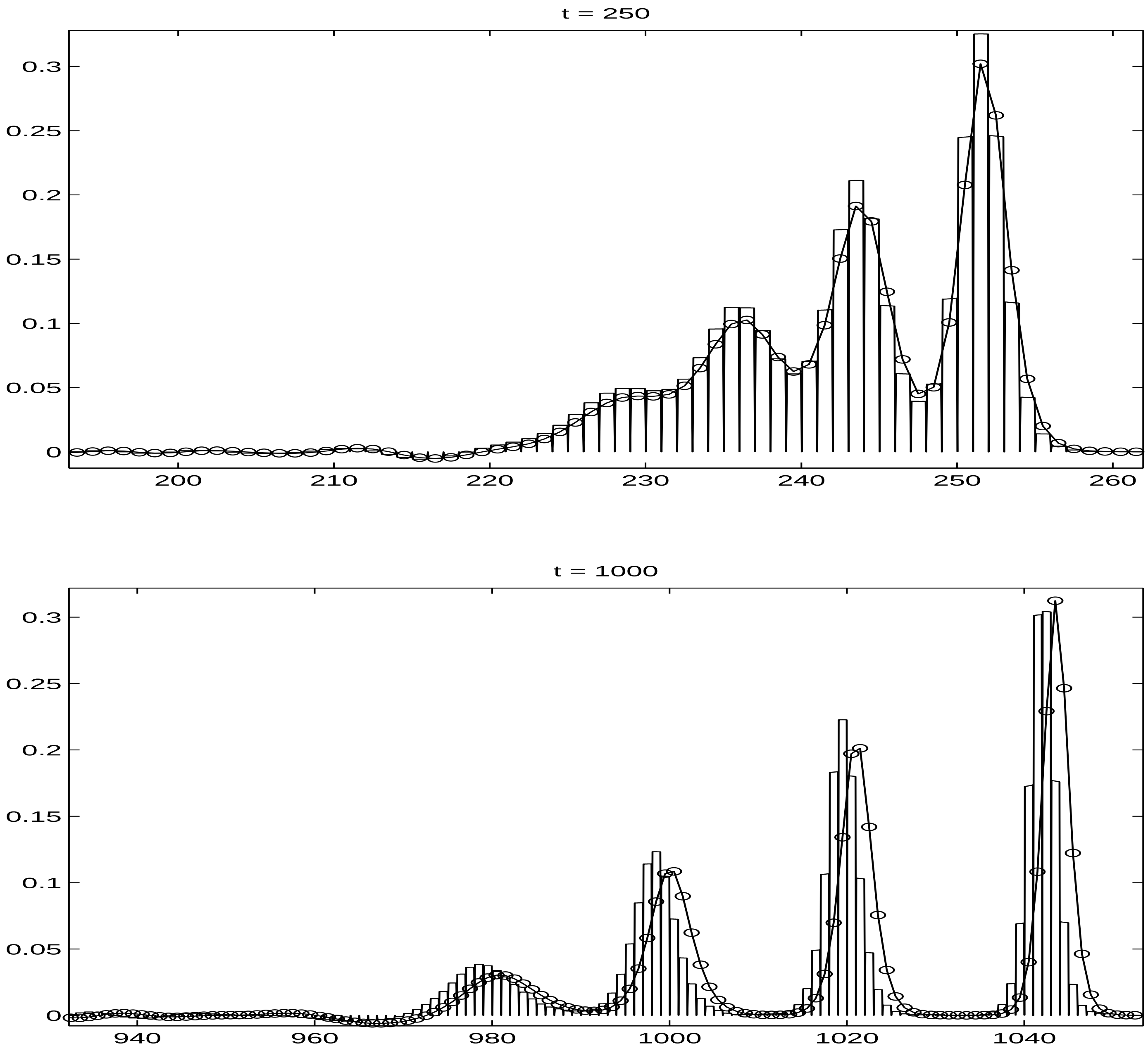
"particle layers:" $\rho_A = O(1/w_A), \;\; K_A \gg 1$

"spring layers:" $\rho_B \ll 1, \;\; K_B = O(1),$

$$\sigma_B(\epsilon) = \exp(K_B \epsilon) - 1$$

Take $c_A, \; c_B \;\; \gg \bar{c}$ so that states equilibrate quickly within layers.

Comparison of layered medium with Toda lattice



Homogenization theory

for long wavelengths (relative to layer width)

$$\begin{aligned} \epsilon_t - u_x &= 0 \\ \rho(x) u_t - \sigma_x &= 0 \end{aligned}$$

Rewrite as equations for σ and u since these are continuous.

$$\sigma_t = K(x)\epsilon_t$$

Obtain

$$\frac{1}{K(x)}\sigma_t - u_x = 0$$

$$\rho(x) u_t - \sigma_x = 0$$

Homogenization theory

Average over distance long relative to variation in K, ρ,
but short relative to variation in u, σ:

$$\left\langle \frac{1}{K(x)} \right\rangle \sigma_t - u_x = 0 + \mathcal{O}(\delta^2)$$

$$\langle \rho(x) \rangle \, u_t - \sigma_x = 0 + \mathcal{O}(\delta^2)$$

or

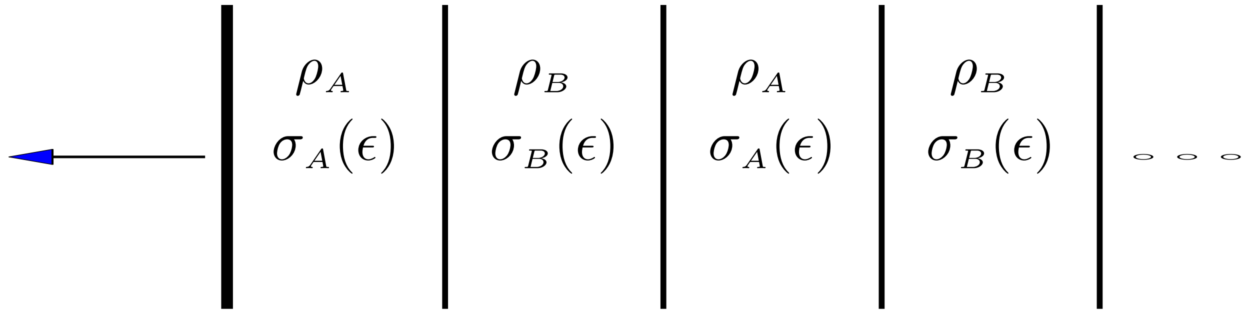
$$\sigma_t - \hat{K} u_x = 0 + \mathcal{O}(\delta^2)$$

$$\bar{\rho} u_t - \sigma_x = 0 + \mathcal{O}(\delta^2)$$

where $\bar{\rho} = \langle \rho \rangle$, $\quad \hat{K} = \left\langle K^{-1} \right\rangle^{-1} =$ harmonic average of K.

Wave speeds $\pm\bar{c} = \pm\sqrt{\hat{K}/\bar{\rho}}$.

Linear periodic layered medium



$$\sigma_A = K_A \epsilon, \quad \sigma_B = K_B \epsilon.$$

Averaged parameters: $\bar{\rho} = \langle \rho \rangle = w_A \rho_A + w_B \rho_B$

$$\hat{K} = \left\langle \frac{1}{K} \right\rangle^{-1} = \left(\frac{w_A}{K_A} + \frac{w_B}{K_B} \right)^{-1}$$

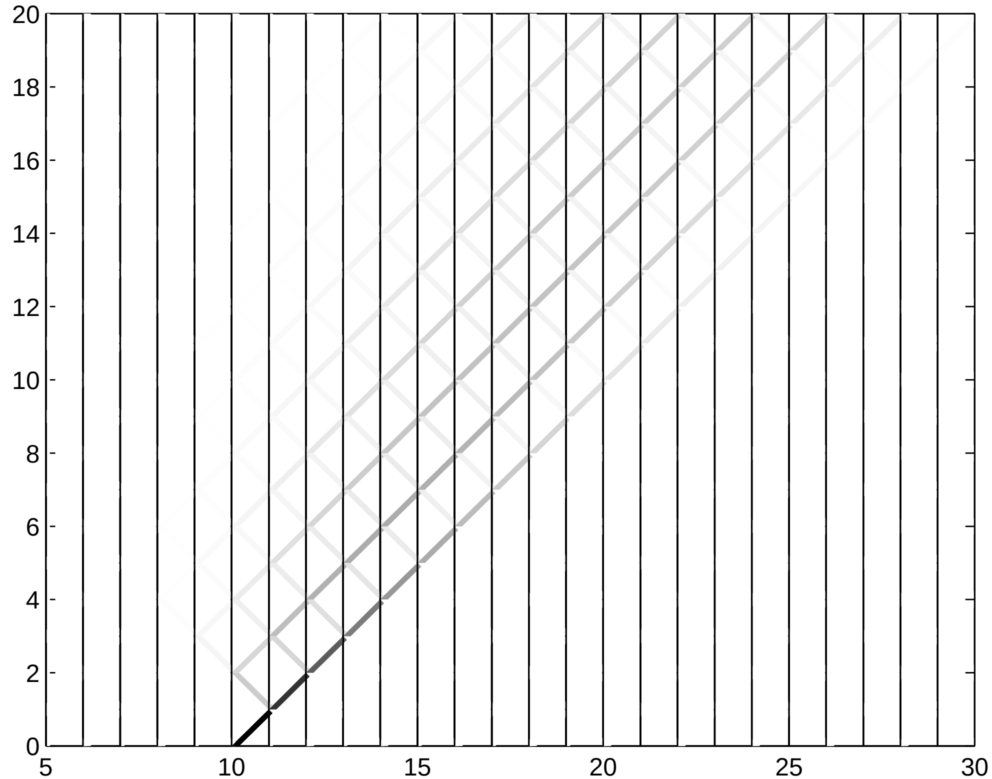
Effective wave speed: $\bar{c} = \sqrt{\hat{K}/\bar{\rho}}.$

Ex: (Santosa & Symes SIAP 51 (1991) p. 984)

$$\rho_A = K_A = 4, \qquad \rho_B = K_B = 1, \qquad w_A = w_B = 1/2.$$

Then $c \equiv 1$ everywhere but $\bar{c} = 0.8$.

Linear periodic layered medium



Also dispersive, with dispersion relation

$$\omega(\xi) = \bar{c}\xi + d\xi^3 + \cdots$$

Ref: Santosa & Symes SIAP 51 (1991) p. 984

Nonlinear homogenization theory

Work of Darryl Yong, based on his thesis with J. Kevorkian.

For the nonlinear stress strain relation

$$\sigma(\epsilon, x) = K(x)\epsilon + \beta K^2(x)\epsilon^2$$

and layers of equal width $\delta/2$, small relative to the wavelength.

$$\begin{aligned}
\bar{\rho} u_t - \sigma_x &= \delta^2 \left(\frac{(\rho_A - \rho_B)(Z_A^2 - Z_B^2)}{24(K_A + K_B)(\rho_A + \rho_B)^2} \right) \sigma_{xxx} + O(\delta^3) \\
\sigma_t - \hat{K} u_x &= 2\hat{K}(\beta\sigma - \beta^2\sigma^2) u_x \\
&\quad + \delta^2 \left(\frac{\hat{K}(K_A - K_B)(Z_A^2 - Z_B^2)}{24(K_A + K_B)^2(\rho_A + \rho_B)} \right) u_{xxx} + O(\delta^3).
\end{aligned}$$

$Z = \sqrt{K\rho} =$ impedance

- $\beta = 0 \implies$ linear result of Santosa-Symes,
- $Z_A = Z_B \implies$ dispersive terms vanish.

Homogenized equations for exponential stress-strain

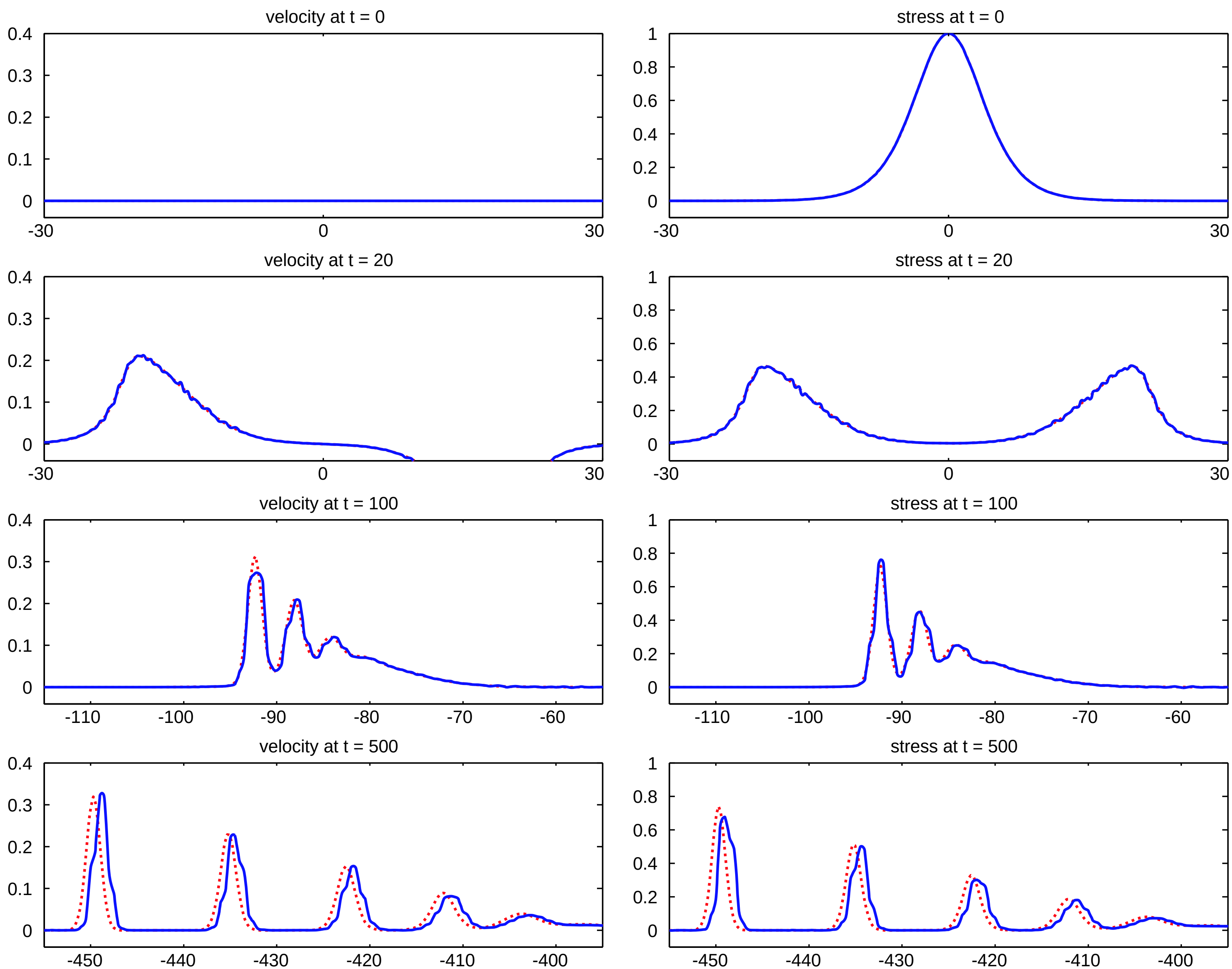
$$\sigma(\epsilon, x) = e^{\epsilon K(x)} - 1$$

$$\rho_A = K_A = 4, \quad \rho_B = K_B = 1, \quad w_A = w_B = 1/2.$$

$$u_t = \frac{2\sigma_x}{5} + \frac{3\delta^2\sigma_{xxx}}{500} + \delta^4\left(\frac{3\sigma_{xxx}\sigma_x^2}{15625(\sigma+1)^2} - \frac{72u_{xx}^2\sigma_x}{15625(\sigma+1)} - \frac{12\sigma_{xxxx}\sigma_x}{15625(\sigma+1)} - \frac{96u_{xx}u_{xxx}}{15625} - \frac{12\sigma_{xx}\sigma_{xxx}}{15625(\sigma+1)} - \frac{357\sigma_{xxxxx}}{1000000}\right) + \mathcal{O}(\delta^6)$$

$$\sigma_t = \frac{8(\sigma+1)u_x}{5} + \delta^2\left(\frac{3(\sigma+1)u_{xxx}}{125} + \frac{3u_{xx}\sigma_x}{50}\right) + \delta^4\left(\frac{48u_x u_{xx}^2}{15625} - \frac{48\sigma_x\sigma_{xx}u_{xx}}{15625(\sigma+1)} - \frac{4761\sigma_{xxx}u_{xx}}{500000} - \frac{72u_{xxx}\sigma_x^2}{15625(\sigma+1)} - \frac{357(\sigma+1)u_{xxxxx}}{250000} - \frac{3543u_{xxxx}\sigma_x}{500000} - \frac{3891u_{xxx}\sigma_{xx}}{500000}\right) + \mathcal{O}(\delta^6).$$

Comparison of homogenized solution to DNS



Summary

- Elastic wave equations in one dimension
- Hyperbolic in homogeneous medium
- Layered medium with impedance mismatch leads to dispersion
- Dispersion plus nonlinearity leads to solitary waves
- Relation to discrete solitons in Toda lattice
- Homogenized equations can be derived

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Many open questions, e.g.,

- Are there exact “solitons” for the layered media equations for particular choice of $\sigma(\epsilon, x)$?
- Behavior with other layer parameters, smooth periodic, random media, ...
- Multidimensional