Chapter 5

CONSERVATION LAWS ON MANIFOLDS

The natural mathematical arena to formulate conservation laws on curved manifolds is that of differential geometry. Ricci developed this branch of mathematics from 1887 to 1896. Subsequent work in differential geometry has made it an indespensible tool for solving problems in mathematical physics. In this chapter we develop the necessary ideas from differential geometry to formulate conservation laws on curved manifolds. We will derive the explicit form of both the spatially-varying flux and geometrically induced source term found in equation (1.4). The chapter also serves to set notational convention for subsequent use. We closely follow the notation of Misner, Thorne, and Wheeler [61] and Chandrasekhar [14].

Differential geometry is a vast field and we only attempt to cover necessary concepts in this chapter. For a complete study the reader is referred to the literature. Good sources for the methods of differential geometry from a physicist's point of view include Nakahara [62], Wald [79], Stewart [75], and Hawking and Ellis [30]. A more mathematical approach can be found in Bishop and Goldberg [8], Lovelock and Rund [51], or Lee [40]. Introductions to the ideas of differential geometry and general relativity can be found in Foster and Nightingale [25] or Schutz [71].

5.1 Manifolds

Differential geometry describes the geometric structure of a curved differentiable manifold, $\mathscr{M}$. A few examples of manifolds where briefly discussed in Chapter 1. For example, a manifold $\mathscr{M}$ may represent the nearly spherical surface of the earth, a four-dimensional curved spacetime in relativity theory, or the two-dimensional surface of an elastic membrane. We develop an intuitive notion of a manifold by the following heuristic definition :

DEFINITION 5.1.1 *A manifold is a set of points that looks locally Euclidean in that this set can be entirely covered by a collection of local coordinate mappings.*

The main thing to notice about this definition of manifolds is that is suggests that it may take multiple coordinate maps to cover $\mathscr{M}$. We have encountered an example of this in Figure 1.1, where the surface of a sphere is covered by six coordinate patches. The use of these six coordinate patches eliminates the polar singularities in the standard Longitude-Latitude coordinate system in favor of more well behaved singularities at the points where three such grids meet.

The main reason that it may be necessary to use multiple coordinate maps to cover a manifold is that the global geometry of $\mathscr{M}$ may be very different from the local geometry near a point. It may be impossible to simply define a local coordinate system in the vicinity of a point and extend it to cover the entire manifold. For example, Figure 5.1 shows the

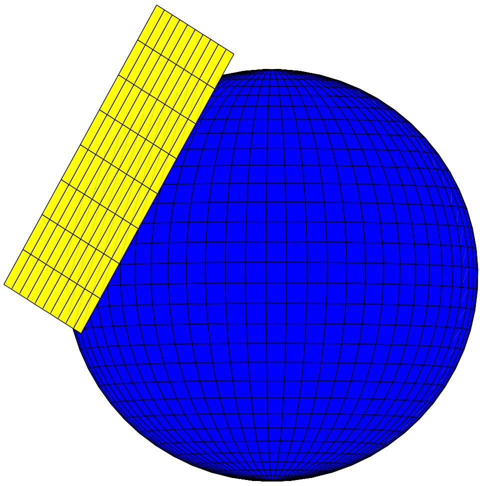

Figure 5.1: The extension of a locally flat coordinate patch tangent to $\mathcal{S}^2$ at a point.

tangent plane at an arbitrary point in $\mathcal{S}^2$. A Cartesian coordinate system on the tangent plane may describe the geometry in a very small region surrounding the point, but it fails to represent the geometry of $\mathcal{S}^2$ on a global scale.

It may be, however, that only a subset $\mathscr{U} \subset \mathscr{M}$ is of interest and that $\mathscr{U}$ can be covered by a single coordinate system. For example, suppose we are not interested in all of $\mathcal{S}^2$, but only the region shown in Figure 5.1 between the Tropic of Capricorn and the Tropic of Cancer. This region can be covered by a coordinate patch defined by

$$\begin{aligned} &\mathcal{X} : \mathscr{U} \subset \mathscr{M} \to \mathcal{U} \subset \mathbb{R}^2 \\ &\left(x^1, x^2\right) = (\theta, \phi)\,, \end{aligned} \tag{5.1}$$

with $\theta \in [\Theta_-, \Theta_+]$ and $\phi \in [0, 2\pi)$, where Θ_- and Θ_+ represent the degrees latitude of the Tropics of Capricorn and Cancer, respectively. If a geophysical flow were to be computed in the coordinate patch $\mathcal{U}$, then only the coordinate map $\mathcal{X}$ would be needed with boundary conditions at $x^1 = \Theta_-$ and $x^1 = \Theta+$, as well as periodic conditions at $x^2 = 0$. In Chapter 8 we calculate the in-fall of a pressure-less gas into a Schwarzschild black hole. In that case, much like the example here, we compute on a subset of the entire manifold for which the coordinates we use (Schwarzschild coordinates) have no singularities. Extending these coordinates to the full manifold, however, would introduce coordinate singularities similar to the polar singularities on $\mathcal{S}^2$ discussed above.

5.2 Tangent Vectors

In flat Euclidean space, a vector $\vec{v}$ is a geometric object that connects two distinct points $\{\mathcal{P}, \mathcal{Q}\}$ and possesses both a magnitude and direction. The magnitude of $\vec{v}$ is simply the

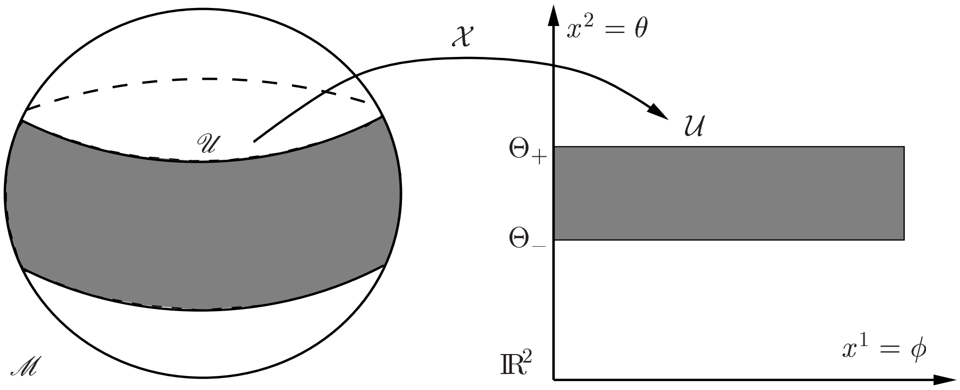

Figure 5.2: A local region $\mathscr{U} \subset \mathscr{M}$ of a 2-sphere is covered by the 2-dimensional coordinate map $\mathcal{X} = (\phi, \theta)$ in $\mathbb{R}^2$.

distance between the two points $|\mathcal{Q} - \mathcal{P}|$ and the direction is determined by connecting $\mathcal{P}$ and $\mathcal{Q}$ by a straight line $\mathcal{L}(\lambda) = \mathcal{P} + \lambda(\mathcal{Q} - \mathcal{P})$ as in Figure 5.3(a).

In a curved manifold, however, it is necessary to modify this flat-space notion of a vector. The concept of a straight line connecting $\mathcal{P}$ and $\mathcal{Q}$ does not even exist. The points can be connected with a curve, but in general there is no unique curve connecting distinct points in $\mathscr{M}$. Therefore, defining the direction and magnitude of a vector based on one such curve is ambiguous. Instead, a vector is defined in terms of the directional derivative tangent to a curve in $\mathscr{M}$ as shown in Figure 5.3(b). Before a precise definition for tangent vectors can be used, the representation of points, functions, and curves on $\mathscr{M}$ must be discussed.

Points in the manifold can be expressed in terms of coordinate functions by using a coordinate map $\mathcal{X} : \mathscr{M} \to \mathbb{R}^{n_d}$. For example, a point, $\mathcal{P} \in \mathscr{M}$ has the coordinate representation

$$\mathcal{X}(\mathcal{P}) = (x^1(\mathcal{P}), \cdots, x^{n_d}(\mathcal{P})), \tag{5.2}$$

where each x^j represent coordinate functions. Functions defined on the manifold can be expressed in the coordinate system as

$$f(\mathcal{X}(\mathcal{P})) = f(x^1(\mathcal{P}), \cdots, x^{n_d}(\mathcal{P})). \tag{5.3}$$

A useful set of curves passing through a point $\mathcal{P} \in \mathscr{M}$ are called coordinate curves and denoted by $C_k(\lambda)$. In terms of the coordinates, these curves can be expressed as

$$\mathcal{X}(C_k(\lambda)) = (x^1(\mathcal{P}), \cdots, \underbrace{x^k(\mathcal{P}) + \lambda}_{k^{\text{th}}\text{ location}}, \cdots, x^{n_d}(\mathcal{P})), \tag{5.4}$$

where all coordinate functions are held constant at $\mathcal{P}$, except for the k^{th} coordinate which varies with λ. A single component of the coordinate representation of $C_k(\lambda)$ passing through $\mathcal{P}$ is expressed as

$$x^j(C_k(\lambda)) = x^j(\mathcal{P}) + \lambda\, \delta^j_k. \tag{5.5}$$

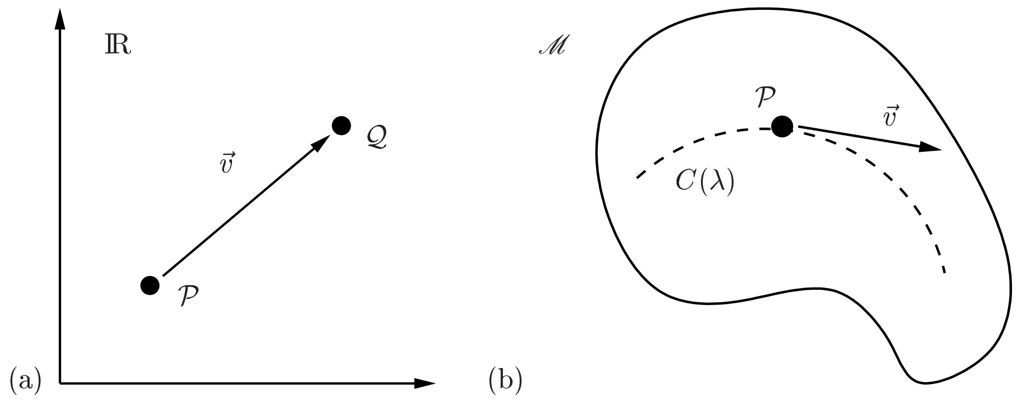

Figure 5.3: (a) A flat-space vector has both magnitude and direction defined by a parameterized line connecting two points $\mathcal{P}$ and $\mathcal{Q}$. (b) In curved space a vector is defined by a point-wise directional derivative operation on smooth functions.

The coordinate expressions described above are useful when defining a tangent vector on $\mathscr{M}$.

DEFINITION 5.2.1 *Let $\mathscr{F}$ be the set of all c^∞ functions defined on $\mathscr{M}$. A* **tangent vector** *at the point $\mathcal{P} = C(\lambda_0) \in \mathscr{M}$ along a curve $C(\lambda)$ is a map $\vec{v} : \mathscr{F} \to \mathbb{R}$ measuring the rate of change of a function $f \in \mathscr{F}$ along the curve $C(\lambda)$ at the point $\mathcal{P}$.*

Vectors that are tangent to coordinate curves are called canonical basis vectors and are denoted by $\vec{\partial}_{(k)}$. According to Definition 5.2.1, the action of $\vec{\partial}_{(k)}$ on a function $f(x^j(C_k(\lambda))) \in \mathscr{F}$ at $\mathcal{P}$ is the partial derivative

$$\begin{aligned}
\vec{\partial}_{(k)} f &= \frac{d}{d\lambda}\Big(f(x^j(C_k(\lambda)))\Big)\Big|_{\mathcal{P}} \\
&= \sum_{j=1}^{n_d} \frac{\partial}{\partial x^j} f \, \frac{d}{d\lambda}\left(x^j(C_k(\lambda))\right)\Big|_{\mathcal{P}} \qquad (5.6)\\
&= \frac{\partial}{\partial x^k} f,
\end{aligned}$$

where we have used the fact that according to the coordinates (5.5), the tangent to the k^{th} coordinate curve (5.4) at the point $\mathcal{P}$ has a one in the k^{th} component and zeros in all the other components since $d(x^j(C_k(\lambda)))/d\lambda = \delta^j_k$.

Now consider the coordinate representation of the general curve shown in Figure 5.3(b)

$$\mathcal{X}(C(\lambda)) = \left(x^1(\lambda), \cdots, x^{n_d}(\lambda)\right), \qquad (5.7)$$

where, as before, $x^k(\lambda)$ represents the k^{th}-parameterized coordinate function of the curve in the n_d-dimensional coordinate system $\mathcal{X}$. The direction tangent to C at the point $\mathcal{P}$ is now

$$\frac{d}{d\lambda}\Big(\mathcal{X}(C(\lambda))\Big)\Big|_{\mathcal{P}} = \left(\frac{d}{d\lambda}x^1(\lambda), \cdots, \frac{d}{d\lambda}x^{n_d}(\lambda)\right)\Bigg|_{\lambda=\lambda_0}$$
$$\equiv \left(\dot{x}^1, \cdots, \dot{x}^{n_d}\right)\Big|_{\lambda=\lambda_0}. \tag{5.8}$$

A vector $\vec{v}$ acts on a function restricted to this general curve, $f(C(\lambda)) = f(x^1(\lambda), \cdots, x^{n_d}(\lambda))$, to give the directional derivative

$$\begin{aligned} \vec{v}\, f &= \frac{d}{d\lambda}\Big(f(x^1(\lambda), \cdots, x^{n_d}(\lambda))\Big)\Big|_{\mathcal{P}=C(\lambda_0)} \\ &= \sum_{j=1}^{n_d}\left(\frac{dx^j}{d\lambda}\frac{\partial}{\partial x^j}\left(f(x^1, \cdots, x^{n_d})\right)\right)\Bigg|_{\mathcal{P}} \\ &\equiv \dot{x}^j \vec{\partial}_{(j)}\, f|_{\mathcal{P}}, \end{aligned} \tag{5.9}$$

where, in the last step, we have used equation (5.6) and the convention that repeated indices are to be summed over.

Equation (5.9) serves to define the vector $\vec{v}$ as a linear combination of the set $\{\vec{\partial}_{(k)}\}$,

$$\vec{v} = v^k\, \vec{\partial}_{(k)}, \tag{5.10}$$

with components defined by $v^k \equiv \dot{x}^k$. In fact, the set $\{\vec{\partial}_{(k)}\}$ represents the basis for the n_d-dimensional vector space of all tangent vectors at the point $\mathcal{P} \in \mathscr{M}$. We denote this tangent space by $T_{\mathscr{M}}(\mathcal{P})$ and elements $\vec{v} \in T_{\mathscr{M}}(\mathcal{P})$ are called *tangent vectors* (or contravariant vectors). It is easy to show that the set $\{\vec{\partial}_{(k)}\}$ is linearly independent. Suppose $0 = v^k\, \vec{\partial}_{(k)}$, then it is also true that $0 = v^k\, \vec{\partial}_{(k)}[x^j] = v^k \delta^j_k = v^j$ demonstrating that they are linearly independent. Equation (5.9) demonstrates that any general vector in $T_{\mathscr{M}}(\mathcal{P})$ can be represented by this set of basis vectors so that $\mathrm{span}(\{\vec{\partial}_{(k)}\}) = T_{\mathscr{M}}(\mathcal{P})$. Figure 5.4 depicts the tangent space for a 2-dimensional manifold. The figure shows two coordinate curves C_1 and C_2 that go through the point $\mathcal{P}$ and generate the basis vectors $\{\vec{\partial}_{(1)}, \vec{\partial}_{(2)}\}$. It also shows a general curve C generating the vector $\vec{v}$, a linear combination of the basis vectors. One can imagine generating all other vectors in $T_{\mathscr{M}}(\mathcal{P})$ by choosing different curves $C(\lambda) \subset \mathscr{M}$ with different parameterizations that go through $\mathcal{P}$.

It is common to display the components of a vector as elements of an n_d-column vector, $\left[v^k\right] = (v^1, v^2, \cdots, v^{n_d})^T$, where the superscript T denotes a transpose. If we also define the $n_d \times n_d$ matrix E

$$E = \left(\left[\vec{\partial}_{(1)j}\right]\Big|\left[\vec{\partial}_{(2)j}\right]\Big|\cdots\Big|\left[\vec{\partial}_{(n_d)j}\right]\right) = \begin{pmatrix} \vec{\partial}_{(1)1} & \vec{\partial}_{(2)1} & \cdots & \vec{\partial}_{(n_d)1} \\ \vec{\partial}_{(1)2} & \vec{\partial}_{(2)2} & \cdots & \vec{\partial}_{(n_d)2} \\ \vdots & \vdots & \ddots & \vdots \\ \vec{\partial}_{(1)n_d} & \vec{\partial}_{(2)n_d} & \cdots & \vec{\partial}_{(n_d)n_d} \end{pmatrix}, \tag{5.11}$$

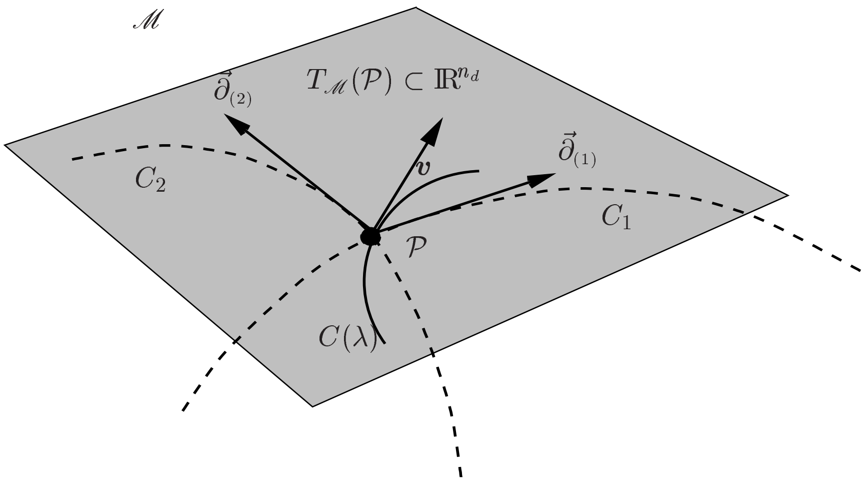

Figure 5.4: Tangent space $T_{\mathscr{M}}(\mathcal{P})$ at $\mathcal{P}$.

whose k^{th} column is the basis vector $\left[\vec{\partial}_{(k)j}\right]$ in matrix form with $j \in \{1, 2, \cdots, n_d\}$, then the vector (5.10) can be represented by

$$\vec{v} = E\left[v^k\right]. \tag{5.12}$$

We make use of this notation in much of what follows.

Since $T_{\mathscr{M}}(\mathcal{P})$ is an n_d-dimensional vector space, its basis is not unique. In fact, the basis can be any linearly independent set of vectors that span $T_{\mathscr{M}}(\mathcal{P})$. We can form a new basis by taking linear combinations of the canonical basis

$$\bar{E} = E\left[\Pi^j_{\bar{k}}\right], \tag{5.13}$$

so that the new set of basis vectors $\{\vec{e}_{(\bar{k})}\}$ make up the columns of $\bar{E}$. Each new basis vector can be written

$$\vec{e}_{(\bar{k})} = \Pi^j_{\bar{k}}\,\vec{\partial}_{(j)}, \tag{5.14}$$

where $\Pi^j_{\bar{k}}$ is the element in the j^{th} row and the $\bar{k}^{\text{th}}$ column of the transformation matrix $\left[\Pi^j_{\bar{k}}\right]$. Therefore, the $\bar{k}^{\text{th}}$ column of $\left[\Pi^j_{\bar{k}}\right]$ contains the coefficients for the linear combination (5.14).

A change of basis vectors will result in a different set of vector components $\left[v^{\bar{k}}\right]$. We discussed in Section 5.1 that it is often necessary to make use of multiple coordinate maps (and therefore multiple sets of basis vectors) when working on curved manifolds. Because of this, it is important to keep in mind that a tangent vector is a geometric object independent of any basis onto which it is projected. Using the notation from above, this can be expressed as

$$\vec{v} = E\left[v^k\right] = \bar{E}\left[v^{\bar{k}}\right]. \tag{5.15}$$

We can use this together with equation (5.13) to conclude that a transformation of basis vectors induces the following transformation of tangent vector components,

$$\left[v^j\right] = \left[\Pi^j_{\bar{k}}\right] \left[v^{\bar{k}}\right], \tag{5.16}$$

$$\left[v^{\bar{k}}\right] = \left[\Pi^j_{\bar{k}}\right]^{-1} \left[v^j\right] \equiv \left[\Pi^{\bar{k}}_j\right] \left[v^j\right]. \tag{5.17}$$

Example 5.2.1 Consider an example where we scale the canonical basis in $\mathbb{R}^2$ to be the standard Cartesian basis

$$E = (\vec{e}_{(1)}|\vec{e}_{(2)}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.18}$$

We wish to express the components of the unit vector $\left[v^k\right] = \frac{1}{\sqrt{2}}(1,1)^T$ in a new basis that is a linear combination of the Cartesian basis vectors $\vec{e}_{(1)}$ and $\vec{e}_{(2)}$. The linear combination will be of the form

$$\begin{aligned} \vec{e}_{(\bar{1})} &= \alpha\,\vec{e}_{(1)} + \beta\,\vec{e}_{(2)} \\ \vec{e}_{(\bar{2})} &= -\alpha\,\vec{e}_{(1)} + \beta\,\vec{e}_{(2)}, \end{aligned} \tag{5.19}$$

where α, $\beta \neq 0$. Notice that the basis remains orthogonal if $|\alpha| = |\beta|$ since $\vec{e}_{(\bar{1})} \cdot \vec{e}_{(\bar{2})} = \beta^2 - \alpha^2$. In fact, the transformation represents a rotation (and possibly a scaling) if $\alpha = \beta$. In equations (5.13) and (5.14), the k^{th} column of $\left[\Pi^j_{\bar{k}}\right]$ represents the coefficients in the linear combination of $\vec{e}_{(j)}$ that are needed for the new basis vector $\vec{e}_{(\bar{k})}$. In this example the transformation matrix and its inverse take the simple form

$$\left[\Pi^j_{\bar{k}}\right] = \begin{pmatrix} \alpha & -\alpha \\ \beta & \beta \end{pmatrix} \qquad \text{and} \qquad \left[\Pi^j_{\bar{k}}\right]^{-1} = \frac{1}{2\alpha\beta} \begin{pmatrix} \beta & \alpha \\ -\beta & \alpha \end{pmatrix}. \tag{5.20}$$

The transformed matrix of basis vectors, according to equation (5.13), can be written

$$\bar{E} = \left(\vec{e}_{(\bar{1})}|\vec{e}_{(\bar{2})}\right) = \begin{pmatrix} \alpha & -\alpha \\ \beta & \beta \end{pmatrix}, \tag{5.21}$$

and the tangent vector components in this new basis $\{\vec{e}_{(\bar{k})}\}$ can be written using equation (5.17) as

$$\left[v^{\bar{k}}\right] = \frac{1}{2\sqrt{2}\alpha\beta} \begin{pmatrix} \alpha + \beta \\ \alpha - \beta \end{pmatrix}. \tag{5.22}$$

Notice that if $\alpha = \beta$ there is never a component in the direction of $\vec{e}_{(\bar{2})}$. In this case, as discussed above, the matrix $\left[\Pi^j_{\bar{k}}\right]$ always rotates the Cartesian basis through an angle of 45°. Since the vector $\left[v^k\right] = 1/\sqrt{2}(1,1)^T$ is along the 45° line, this vector will be entirely aligned with $\vec{e}_{(\bar{1})}$ in the transformed basis, leaving no component in the orthogonal direction of $\vec{e}_{(\bar{2})}$. The transformation discussed here with $\alpha = \beta$ is shown in Figure 5.5(b). The basis vectors are rotated by 45° so that the vector is along $\vec{e}_{(\bar{1})}$.

Since $\alpha = \beta$, but $\alpha \neq 1/\sqrt{2}$, the Cartesian basis is scaled by α as well as rotated by 45°. In this case, notice that the first component in $\left[v^{\bar{k}}\right]$ is inversely scaled with α so that $1/(\sqrt{2}\alpha)$ and the magnitude of the vector remains the same. The components of $\vec{v}$ in this new basis are shown in Figure 5.5(b).

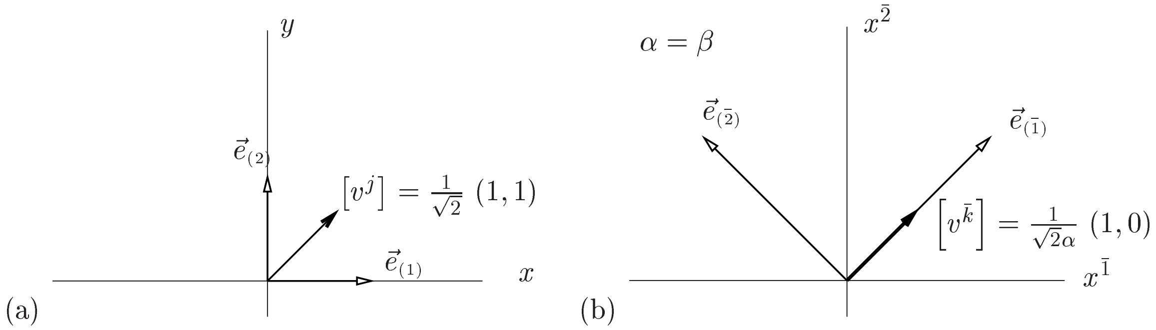

Figure 5.5: (a) The vector $\vec{v} = 1/\sqrt{2}(1,1)$ represented in a standard Cartesian basis. (b) The same vector represented in the new basis $\{\vec{e}_{(\bar{k})}\}$ that has been rotated through 45° and scaled by α. The component of $\vec{v}$ must be scaled by $1/\alpha$ to keep the magnitude the same.

5.2.1 Coordinate transformations

A special case of the transformation of basis vectors is that induced by a coordinate transformation of the form,

$$x^j \to x^j(x^{\bar{k}}). \tag{5.23}$$

The chain rule of partial derivatives requires that if $f \in \mathscr{F}$ then

$$\vec{\partial}_{(\bar{k})}[f] = \frac{\partial x^j}{\partial x^{\bar{k}}} \vec{\partial}_{(j)}[f], \tag{5.24}$$

from which is follows that the components of the transformation matrix $\left[\Pi^j_{\bar{k}}\right]$ have the form

$$\Pi^j_{\bar{k}} = \frac{\partial x^j}{\partial x^{\bar{k}}}. \tag{5.25}$$

A basis that is generated by transforming a canonical basis under a coordinate transformation (5.23) is called a *coordinate basis*. In a coordinate basis, the vector component v^k is a number that represents the projection of $\vec{v}$ along the coordinate curve C_k.

Example 5.2.2 A coordinate transformation that gives rise to the transformed basis that we computed in Example 5.2.1 and show in Figure 5.5(b) is

$$x^1(x^{\bar{1}}, x^{\bar{2}}) = \alpha\,(x^{\bar{1}} - x^{\bar{2}}) \tag{5.26}$$

$$x^2(x^{\bar{1}}, x^{\bar{2}}) = \beta\,(x^{\bar{1}} + x^{\bar{2}}). \tag{5.27}$$

The transformation matrix has the form

$$\left[\Pi^j_{\bar{k}}\right] = \begin{pmatrix} \frac{\partial x^1}{\partial x^{\bar{1}}} & \frac{\partial x^1}{\partial x^{\bar{2}}} \\ \frac{\partial x^2}{\partial x^{\bar{1}}} & \frac{\partial x^2}{\partial x^{\bar{2}}} \end{pmatrix} = \begin{pmatrix} \alpha & -\alpha \\ \beta & \beta \end{pmatrix}, \tag{5.28}$$

just as we found in (5.20).

5.3 Cotangent vectors

Associated with $T_{\mathscr{M}}(\mathcal{P})$ is a dual vector space $T^*_{\mathscr{M}}(\mathcal{P})$. Elements $\tilde{w} \in T^*_{\mathscr{M}}(\mathcal{P})$ are called **cotangent vectors** (also called covariant vectors, co-vectors, or one-forms) and are linear maps $\tilde{w} : T_{\mathscr{M}}(\mathcal{P}) \to \mathrm{I\!R}$. A basis $\{\tilde{e}^{(k)}\}$ for $T^*_{\mathscr{M}}(\mathcal{P})$, associated with the basis $\{\vec{e}_{(k)}\}$ on $T_{\mathscr{M}}(\mathcal{P})$, maps any tangent vector $\vec{v} = v^k\,\vec{e}_{(k)}$ to its components,

$$\tilde{e}^{(j)}(\vec{v}) = \langle \tilde{e}^{(j)}, v^i\,\vec{e}_{(i)} \rangle = v^j \quad j \in \{1, 2, \cdots, n_d\}. \tag{5.29}$$

From this last equation (and the fact that the map is linear) it follows that

$$\tilde{e}^{(j)}(\vec{e}_{(i)}) = \langle \tilde{e}^{(j)}, \vec{e}_{(i)} \rangle = \delta^j_i. \tag{5.30}$$

The fact that the set $\{\tilde{e}^{(j)}\}$ constitutes a basis for the cotangent space follows if a general cotangent vector can be expressed as a linear combination of this set, and if they are linearly independent. To verify the former we define w_j to be the number to which the co-vector $\tilde{w}$ maps the basis vector $\vec{e}_{(j)}$ so that

$$\tilde{w}(\vec{e}_{(j)}) = \langle \tilde{w}, \vec{e}_{(j)} \rangle \equiv w_j, \tag{5.31}$$

and let $\vec{v} \in T_{\mathscr{M}}(\mathcal{P})$ be an arbitrary tangent vector. Using equation (5.30) and our definition of w_j we can write

$$\begin{aligned} \tilde{w}(\vec{v}) = \langle \tilde{w}, v^k\,\vec{e}_{(k)} \rangle &= w_k\,v^k \\ &= w_k \langle \tilde{e}^{(k)}, v^j\,\vec{e}_{(j)} \rangle \\ &= \langle w_k\,\tilde{e}^{(k)}, \vec{v} \rangle. \end{aligned} \tag{5.32}$$

Since this last equality is true for an arbitrary tangent vector, $\vec{v}$, it follows that

$$\tilde{w} = w_k\,\tilde{e}^{(k)}, \tag{5.33}$$

which demonstrates that $\mathrm{span}(\{\tilde{e}^{(k)}\}) = T^*_{\mathscr{M}}(\mathcal{P})$. Furthermore, if $w_k\,\tilde{e}^{(k)} = 0$, linear independence of the set $\{\tilde{e}^{(k)}\}$ follows because it is then also true that $w_k \langle \tilde{e}^{(k)}, \vec{e}_{(j)} \rangle = w_j = 0$. The set $\{\tilde{e}^{(k)}\}$, therefore, provides a basis for the cotangent space that will be referred to as the *co-basis*. In fact, the basis $\{\vec{e}_{(i)}\}$ and the co-basis $\{\tilde{e}^{(j)}\}$ are dual bases of the tangent and cotangent spaces, respectively.

The differential $\tilde{df}$ of any function $f \in \mathscr{F}$ is a one form whose action on the tangent vector $\vec{v}$ is defined by $\tilde{df}(\vec{v}) \equiv \vec{v}[f] \in \mathrm{I\!R}$. Using (5.9) and the fact that $v^k = \dot{x}^k$ we can write

$$\tilde{df} = \frac{\partial f}{\partial x^k} \tilde{dx}^{(k)}, \tag{5.34}$$

so that the $\tilde{dx}^{(k)}$ form a the canonical basis for $T^*_{\mathscr{M}}(\mathcal{P})$. The cotangent basis $\{\tilde{dx}^{(k)}\}$ is naturally dual to the canonical tangent basis $\{\vec{\partial}_{(k)}\}$ since

$$\tilde{dx}^{(k)}(\vec{\partial}_{(j)}) = \frac{\partial}{\partial x^j} x^k = \delta^k_j. \tag{5.35}$$

A general co-basis $\{\tilde{e}^{(k)}\}$ may be the result of a transformation of a canonical co-basis.

It is often useful to write the components of a co-vector in matrix notation as we did for tangent vectors in Section 5.2. Cotangent vector components, however, are most naturally written as an n_d-row vector $[w_k] = (w_1, w_2, \cdots, w_{n_d})$. Using this matrix notation, a co-vector can be written

$$\tilde{w} = [w_k]\ E_*, \tag{5.36}$$

where we have defined

$$E_* = \begin{pmatrix} \left[\tilde{dx}^{(1)j}\right] \\ \hline \left[\tilde{dx}^{(2)j}\right] \\ \hline \vdots \\ \hline \left[\tilde{dx}^{(n_d)j}\right] \end{pmatrix} = \begin{pmatrix} \tilde{dx}^{(1)1} & \tilde{dx}^{(1)2} & \cdots & \tilde{dx}^{(1)n_d} \\ \tilde{dx}^{(2)1} & \tilde{dx}^{(2)2} & \cdots & \tilde{dx}^{(2)n_d} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{dx}^{(n_d)1} & \tilde{dx}^{(n_d)2} & \cdots & \tilde{dx}^{(n_d)n_d} \end{pmatrix}. \tag{5.37}$$

It is the k^{th} row that represents $\tilde{e}^{(k)}$ in E_*.

In Section 5.2 we derived the transformation of tangent vector components under the influence of the change of basis (5.13). Such a change of basis will induce a transformation of the co-basis since $E_* = E^{-1}$ and $\bar{E}_* = \bar{E}^{-1}$ are required by the duality of the basis and co-basis. Using this in the transformation (5.13) and rearranging we find

$$\bar{E}_* = \left[\Pi^j_{\bar{k}}\right]^{-1} E_*. \tag{5.38}$$

The co-vectors, like the tangent vectors, are geometric objects that are independent of the co-basis so that

$$\tilde{w} = [w_j]\ E_* = [w_{\bar{k}}]\ \bar{E}_*. \tag{5.39}$$

Using this last equation together with the co-basis transformation (5.38), we can write the transformation rule for the components of a cotangent vector

$$[w_j] = [w_{\bar{k}}] \left[\Pi^j_{\bar{k}}\right]^{-1} \equiv [w_{\bar{k}}] \left[\Pi^{\bar{k}}_j\right], \tag{5.40}$$

$$[w_{\bar{k}}] = [w_j] \left[\Pi^j_{\bar{k}}\right]. \tag{5.41}$$

Notice that the transformation to to barred basis involves $\left[\Pi^j_{\bar{k}}\right]$, and not the inverse as it did for tangent vector components in (5.17).

5.3.1 Coordinate transformations

Under a coordinate transformation of the form (5.23) discussed in Section 5.2.1, a canonical co-basis is transformed according to

$$\tilde{dx}^{(\bar{k})} = \frac{\partial x^{\bar{k}}}{\partial x^j}\, \tilde{dx}^{(j)}. \tag{5.42}$$

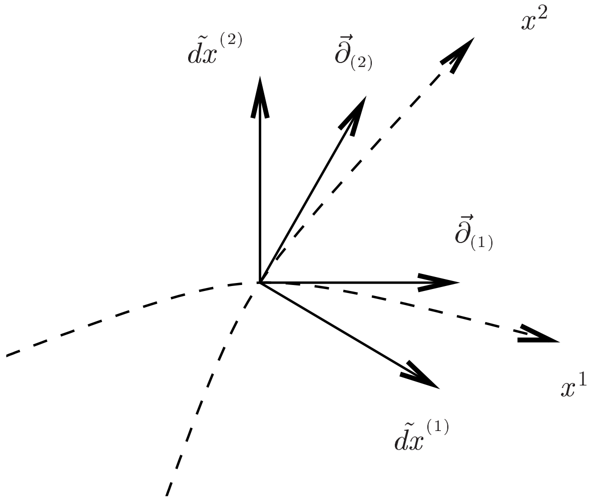

Figure 5.6: The basis $\{\vec{e}_{(i)}\}$ is tangent to the coordinate curves C_1 and C_2 at the point of intersection. The co-basis $\{\tilde{e}^{(i)}\}$, on the other hand, is perpendicular to the opposite coordinate curves.

The last equation makes use of the inverse transformation

$$x^{\bar{k}} \to x^{\bar{k}}(x^j), \tag{5.43}$$

instead of the transformation (5.23). This is the reason that the components of cotangent vectors transform using the inverse transformation matrix.

The duality of a coordinate basis and its co-basis has a geometric interpretation that is shown in Figure 5.6. The basis vectors were constructed based on the direction tangent to the coordinate curves C_k, and therefore point along coordinate lines. Figure 5.6 shows two such coordinate lines and the canonical tangent basis $\{\vec{\partial}_{(1)}, \vec{\partial}_{(2)}\}$ at the intersection point. Also shown are the co-basis vectors $\{\tilde{dx}^{(1)}, \tilde{dx}^{(2)}\}$ which, because of duality, point in a direction perpendicular to the opposite coordinate curve. That is, $\tilde{dx}^{(1)}$ is perpendicular to the tangent of C_2 and $\tilde{dx}^{(2)}$ is perpendicular to the tangent of C_1 at the point of intersection as shown in the figure.

5.4 Tensors

In the previous two sections we introduced tangent vectors and cotangent vectors. A tensor of type $\binom{r}{s}$ is a linear map $\overleftrightarrow{T} : \otimes^r T^*_{\mathscr{M}}(\mathcal{P}) \otimes^s T_{\mathscr{M}}(\mathcal{P}) \to \mathrm{I\!R}$, where the symbols $\otimes^r$ and $\otimes^s$ indicate that the tensor takes r cotangent vectors, and s tangent vectors as arguments. Since tangent vectors map cotangent vectors to real numbers, they are type $\binom{1}{0}$ tensors. Similarly, cotangent vectors are type $\binom{0}{1}$ tensors since they map tangent vectors to $\mathrm{I\!R}$.

The components of a type $\binom{r}{s}$ tensor may be displayed by projecting it onto a basis and its dual co-basis such that

$$\overleftrightarrow{T} = T^{j_1 \cdots j_r}{}_{k_1 \cdots k_s} \vec{e}_{(j_1)} \otimes \cdots \otimes \vec{e}_{(j_r)} \otimes \tilde{e}^{(k_1)} \otimes \cdots \otimes \tilde{e}^{(k_s)}. \tag{5.44}$$

Elements of the basis and its co-basis can be used (once again using duality) to map any tensor to its components. For example, the components of a type $\binom{2}{0}$ tensor $\overleftrightarrow{T}$ can be computed using two elements of the co-basis as

$$\overleftrightarrow{T}(\tilde{e}^{(i)}, \tilde{e}^{(j)}) = T^{ij}. \tag{5.45}$$

Similarly, the components of a type $\binom{0}{2}$ tensor are computed using two elements of the basis as

$$\overleftrightarrow{T}(\vec{e}_{(i)}, \vec{e}_{(j)}) = T_{ij}, \tag{5.46}$$

and components of a type $\binom{1}{1}$ tensor are computed using an element of both the basis and co-basis as

$$\overleftrightarrow{T}(\tilde{e}^{(i)}, \vec{e}_{(j)}) = T^i_j. \tag{5.47}$$

Multidimensional systems for which momentum is conserved have conservation laws with flux functions that are type $\binom{2}{0}$ symmetric tensors. We discuss such conservation laws in Section 5.8.

Components of the tensor defined in (5.44), like tangent vector components in Section 5.2 and cotangent vector components in Section 5.3, can be written in terms of different sets of basis and co-basis vectors. In this particular case there would be r components that transform as tangent vector components, and s components that transform as cotangent components.

In the next section the metric tensor, $\overleftrightarrow{h}$ is introduced. This is a second rank tensor often expressed in terms of its cotangent components h_{ij}. In matrix form this tensor can be expressed in terms of two bases

$$\overleftrightarrow{h} = [E_*]^T \; [h_{ij}] \; [E_*] = \left[\bar{E}_*\right]^T \; \left[h_{\bar{i}\bar{j}}\right] \; \left[\bar{E}_*\right], \tag{5.48}$$

as we saw for both tangent and cotangent vectors. Using the transformation properties in (5.38) for the co-basis, the cotangent components of this tensor transform according to the following:

$$[h_{ij}] = \left[\Pi^i_{\bar{k}}\right]^{-T} \; [h_{\bar{k}\bar{l}}] \; \left[\Pi^j_{\bar{l}}\right]^{-1} \tag{5.49}$$

$$[h_{\bar{k}\bar{l}}] = \left[\Pi^i_{\bar{k}}\right]^{T} \; [h_{ij}] \; \left[\Pi^j_{\bar{l}}\right]. \tag{5.50}$$

If we were interested in the transformation of the tangent components of a tensor, then the transformation (5.13) would be used.

5.5 The Metric Tensor

The metric tensor
$$\overset{\leftrightarrow}{h} = h_{ij}\, \tilde{e}^{(i)} \otimes \tilde{e}^{(j)} \tag{5.51}$$
is a symmetric type $\binom{0}{2}$ tensor that provides a measure of length on $\mathscr{M}$. The path length along a curve $C(\lambda)$ in $\mathscr{M}$ from $C(a)$ to $C(b)$ can be written

$$L = \int_a^b \left| h_{ij}\, \dot{x}^i(\lambda)\, \dot{x}^j(\lambda) \right|^{1/2} d\lambda, \tag{5.52}$$

where $x^k(\lambda)$ represents the coordinate parameterization of the curve. The integrand of the last integral suggests defining the infinitesimal interval dl between neighboring points of the manifold such that
$$dl^2 = h_{ij}\, dx^i\, dx^j, \tag{5.53}$$
which is consistent with equation (5.52). In the previous equation dx^i represents a displacement along the i^{th} coordinate curve, not the i^{th} co-basis $\tilde{dx}^{(i)}$. The metric components h_{ij} are computed using equation (5.45) as

$$h_{ij} = \overset{\leftrightarrow}{h}(\vec{e}_{(i)}, \vec{e}_{(j)}) \equiv \vec{e}_{(i)} \cdot \vec{e}_{(j)}. \tag{5.54}$$

The previous equation defines the *dot product* of two tangent vectors on $\mathscr{M}$ since

$$\begin{aligned} \overset{\leftrightarrow}{h}(\vec{v}, \vec{w}) = \overset{\leftrightarrow}{h}(v^i\, \vec{e}_{(i)}, w^j\, \vec{e}_{(j)}) &= v^i\, w^j\, \overset{\leftrightarrow}{h}(\vec{e}_{(i)}, \vec{e}_{(j)}) \\ &= v^i\, w^j\, h_{ij}, \end{aligned} \tag{5.55}$$

where, in the last equality, we have used equation (5.54) and linearity of $\overset{\leftrightarrow}{h}$.

On a well behaved manifold, the metric tensor must be non-singular. This requirement is equivalent to the requirement that the determinant h of the matrix $[h_{ij}]$ is non-singular at every point of the manifold. It follows that the matrix $[h_{ij}]$ has a unique inverse $[h_{ij}]^{-1} \equiv [h^{ij}]$ so that
$$h_{ij}\, h^{jk} = \delta_i^k. \tag{5.56}$$
The inverse metric is a type $\binom{2}{0}$ tensor with components

$$h^{ij} = \overset{\leftrightarrow}{h}(\tilde{e}^{(i)}, \tilde{e}^{(j)}) \equiv \tilde{e}^{(i)} \cdot \tilde{e}^{(j)}, \tag{5.57}$$

and defines the *dot product* of two cotangent vectors.

Since both the metric tensor and its inverse are non-degenerate, they define an isomorphism $T_{\mathscr{M}}(\mathcal{P}) \leftrightarrow T^*_{\mathscr{M}}(\mathcal{P})$, with
$$v^k = h^{kj}\, w_j \qquad \text{and} \qquad w_k = h_{kj}\, v^j. \tag{5.58}$$

It is, therefore, useful to terminate the notational distinction between the tangent vector and its co-vector. Instead, we distinguish co-vector components from tangent components by using subscripts for the former, and superscripts for the latter as shown in equation (5.58). This is true for tensors as well. Instead of calling a tensor type $\binom{2}{0}$ or type $\binom{0}{2}$, we simply refer to a *second rank* tensor with the isomorphism $T^{ij} \leftrightarrow T^i{}_j \leftrightarrow T_i^{\,j} \leftrightarrow T_{ij}$ understood.

5.6 Derivatives

The components of the basis, co-basis, and metric tensor will be functions of the spatial variables on $\mathscr{M}$. It is therefore necessary to modify the notion of a derivative on such a spatially dependent basis. Consider the gradient of the vector $\vec{v}$

$$\begin{aligned}\vec{\nabla}\vec{v} &= \tilde{e}^{(k)}\nabla_k\left(v^l\,\vec{e}_{(l)}\right)\\ &= \tilde{e}^{(k)}\left[v^l_{,k}\,\vec{e}_{(l)} + v^l\,\nabla_k\vec{e}_{(l)}\right],\end{aligned} \tag{5.59}$$

where we have, once again, made use of the notation that a comma followed by an index denotes the partial derivative, e.g., $\partial v^i/\partial x^j = v^i_{,j}$. Since $\nabla_k\vec{e}_{(l)}$ is a vector, we project it onto the basis

$$\nabla_k\vec{e}_{(l)} = \Gamma^j_{lk}\,\vec{e}_{(j)}, \tag{5.60}$$

where Γ^j_{lk} is the Christoffel symbol that represents the j^{th} component of the derivative of $\vec{e}_{(l)}$ with respect to x^k. The Christoffel symbols can also be written in terms of partial derivatives of metric components

$$\Gamma^j_{lk} = \frac{1}{2}\,h^{jm}\left(h_{mk,l} + h_{lm,k} - h_{lk,m}\right). \tag{5.61}$$

A contraction of the upper index with one of the lower indices can be written as

$$\Gamma^j_{mj} = \frac{1}{2}\,h^{jk}\,h_{jk,m} \tag{5.62}$$

where the fact that repeated indices are *dummy* indices and can be renamed has been used. Using cofactors of matrices, and the fact that $\left[h^{jk}\right]$ is the inverse of $[h_{jk}]$, it can be shown that the derivative of $h = \det(\overleftrightarrow{h})$ is

$$h_{,m} = h\,h^{jk}\,h_{jk,m}. \tag{5.63}$$

Using this in equation (5.62), the contracted Christoffel symbol can be expressed as

$$\Gamma^k_{ik} = \frac{\left(\sqrt{h}\right)_{,k}}{\sqrt{h}}. \tag{5.64}$$

This will be a useful expression throughout this thesis.

Using the Christoffel symbols in equation (5.59), one finds that

$$\vec{\nabla}\vec{v} = \left[v^l_{,k} + v^j\,\Gamma^l_{jk}\right]\tilde{e}^{(k)}\otimes\vec{e}_{(l)}, \tag{5.65}$$

where we have again used the fact that indices that have been summed on are dummy indices and can be renamed. This derivative is called the covariant derivative and is denoted by a semi-colon to distinguish it from the partial derivative. The covariant derivative of a vector is a type $\binom{1}{1}$ tensor whose components can be computed using equation (5.47). For example, the covariant derivative with respect to $\vec{e}_{(m)}$ of the i^{th} component of $\vec{v}$ is

$$\begin{aligned} \vec{\nabla}\vec{v}(\vec{e}_{(m)}, \tilde{e}^{(i)}) &= v^i_{,m} + v^j\,\Gamma^i_{jm} \\ &\equiv v^i_{;m} \end{aligned} \tag{5.66}$$

Since conservation laws depend on the divergence of flux vectors $\vec{\nabla} \cdot \vec{f}(\boldsymbol{q})$, Christoffel symbols will appear in the coordinate version of conservation laws on $\mathscr{M}$. Using equation (5.66) to write the divergence of a vector

$$\begin{aligned} \vec{\nabla} \cdot \vec{v} &= \vec{\nabla}\vec{v}(\vec{e}_{(k)}, \tilde{e}^{(k)}) \\ &= v^k_{,k} + v^j\,\Gamma^k_{ik} \\ &= \frac{1}{\sqrt{h}}\left(\sqrt{h}\,v^k\right)_{,k}, \end{aligned} \tag{5.67}$$

where h is the determinant of the metric tensor and equation (5.64) has been used.

It was mentioned in Section 5.4 that a multidimensional system in which momentum is conserved will have a flux represented by a second rank tensor. Therefore, we will also need to compute the divergence of such a tensor. The new term on the left hand side of equation (5.59) comes from the chain rule. The basis vector in the expression $\vec{v} = v^k\,\vec{e}_{(k)}$ depends on space and must be differentiated. For a second rank tensor $\overleftrightarrow{T} = T^{ij}\,\vec{e}_{(i)} \otimes \vec{e}_{(j)}$, there are two spatially-dependent basis vectors to differentiate using the chain rule. The covariant derivative of a second rank tensor has the form

$$\begin{aligned} \vec{\nabla}\overleftrightarrow{T} &= \tilde{e}^{(k)}\nabla_k\left(T^{lm}\,\vec{e}_{(l)}\,\vec{e}_{(m)}\right) \\ &= \tilde{e}^{(k)}\left[T^{lm}_{,k}\,\vec{e}_{(l)} \otimes \vec{e}_{(m)} + T^l m\,\Gamma^i_{lk}\,\vec{e}_{(i)} \otimes \vec{e}_{(m)} + T^{lm}\,\Gamma^i_{mk}\,\vec{e}_{(l)} \otimes \vec{e}_{(i)}\right] \\ &= \left(T^{lm}_{,k} + T^{im}\,\Gamma^l_{ik} + T^{li}\,\Gamma^m_{ik}\right)\,\tilde{e}^{(k)} \otimes \vec{e}_{(l)} \otimes \vec{e}_{(m)}. \end{aligned} \tag{5.68}$$

Notice that the covariant derivative raises the rank by one. The divergence of a second rank tensor can be computed from (5.68) by taking

$$\begin{aligned} \vec{\nabla} \cdot \overleftrightarrow{T} &= \vec{\nabla}\overleftrightarrow{T}(\vec{e}_{(k)}, \tilde{e}^{(k)}, \cdot) \\ &= \left(T^{km}_{,k} + T^{im}\,\Gamma^k_{ik} + T^{ki}\,\Gamma^m_{ik}\right)\,\vec{e}_{(m)} \\ &= \left(\frac{1}{\sqrt{h}}\left(\sqrt{h}\,T^{km}\right)_{,k} + T^{ki}\,\Gamma^m_{ik}\right)\,\vec{e}_{(m)}, \end{aligned} \tag{5.69}$$

where in this case the divergence is not a simple scalar, but a tangent vector. In fact, the divergence will always be of two ranks less than the gradient tensor because the divergence represents a contraction on two of the indices in the gradient tensor. The gradient of a vector, for example, was shown in equation (5.65) to be a second rank tensor. By contracting on two of the indices for the divergence (5.67), we are left with a scalar. The gradient of a second rank tensor, however, is of rank three (see (5.68)) so that the divergence is of rank one.

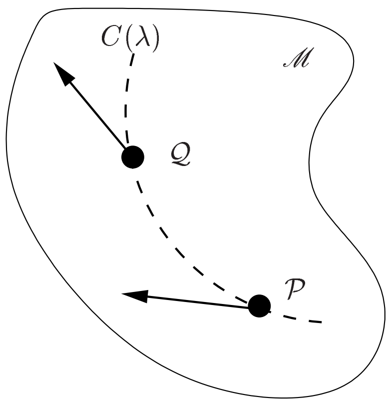

Figure 5.7: The parallel transport of a vector along a curve from a point $\mathcal{P}$ to $\mathcal{Q}$ in the manifold $\mathscr{M}$. Parallel transport with respect to the vector tangent to $C(\lambda)$ leaves the vector covariantly unchanged along the curve.

In the last equality of (5.69) we have again used equation (5.64), but could not eliminate all Christoffel symbols as we could for tangent vectors. This will play a major role in the development of our numerical algorithm in the next chapter.

5.7 Parallel Transport

In Euclidean space, vectors can be moved along arbitrary curves with no change to the components. This is no longer true when we are in a curved space. The best we can do is *parallel transport* a vector along a curve so that it *changes the least.* The *parallel transport* of a vector $\vec{v}$ along a curve with tangent vector $\vec{t} = (\dot{x}^1, \dot{x}^2, \cdots, \dot{x}^{n_d})^T$ is defined by

$$\vec{\nabla}_{\vec{t}}\vec{v} = \dot{x}^k \, v^m_{;k} = 0, \tag{5.70}$$

so that "*changes the least*" means that the vector $\vec{v}$ does not covariantly change along the curve. We will be interested in a discrete version of equation (5.70) in the next chapter so here we investigate the meaning of discrete parallel transport.

Consider two nearby points connected by a curve $C(\lambda)$ such that $\mathcal{P} = C(\lambda)$ and $\mathcal{Q} = C(\lambda + \Delta\lambda)$ as shown in Figure 5.7. The components v^i at $\mathcal{Q}$ can be written in terms of the components at $\mathcal{P}$ using a Taylor expansion

$$v^i(\mathcal{Q}) = v^i(\mathcal{P}) + \Delta\lambda \, \dot{v}^i(\mathcal{P}) + \frac{1}{2}\Delta\lambda^2 \, \ddot{v}^i(\mathcal{P}) + \mathcal{O}\left(\Delta\lambda^3\right), \tag{5.71}$$

where we have dropped any terms of higher order than $\mathcal{O}(\Delta\lambda^2)$. Using the definition of parallel transport (5.70) together with the covariant derivative (5.66), we can write

$$\begin{aligned}
\dot{x}^i\, v^j_{;i} &= \dot{x}^i \left(v^j_{,i} + v^m\, \Gamma^j_{mi}\right) \\
&= \dot{v}^j + v^m\, \dot{x}^i\, \Gamma^j_{mi} \\
&= 0.
\end{aligned} \tag{5.72}$$

The fact that $\vec{v}$ has been parallel transported means that its components satisfy

$$\dot{v}^j = -v^m\, \dot{x}^i\, \Gamma^j_{mi}, \tag{5.73}$$

along the curve. Continuing in this manner we find

$$\ddot{v}^j = \dot{x}^k\, \dot{x}^l\, \Gamma^j_{mk}\, \Gamma^m_{nl}\, v^n - \dot{x}^k\, \dot{x}^l\, \mathcal{G}^j_{mkl}\, v^m - \ddot{x}^k\, \Gamma^j_{mk}\, v^m, \tag{5.74}$$

where we have defined $\mathcal{G}^j_{mkl} \equiv \Gamma^j_{mk,l}$. Using the above expressions for $\dot{v}^j$ and $\ddot{v}^j$ in the expansion (5.71) gives

$$\begin{aligned}
v^j(\mathcal{Q}) = \big[\delta^j_m &- \left(\Delta\lambda\, \dot{x}^i + \frac{1}{2}\Delta\lambda^2\, \ddot{x}^i\right) \Gamma^j_{mi} + \frac{1}{2}\left(\Delta\lambda\, \dot{x}^k\right)\left(\Delta\lambda\, \dot{x}^l\right) \Gamma^j_{nk}\, \Gamma^n_{ml} \\
&- \frac{1}{2}\left(\Delta\lambda\, \dot{x}^k\right)\left(\Delta\lambda\, \dot{x}^l\right) \mathcal{G}^j_{mkl}\big]\, v^m(\mathcal{P}) + \mathcal{O}(\Delta\lambda^3).
\end{aligned} \tag{5.75}$$

This last equation is written in terms of the components of the tangent vector $\vec{t}$, and the parameter λ. It is usually the case that we work directly with the coordinates. Therefore, we relate the change in coordinates between $\mathcal{P}$ and $\mathcal{Q}$ with the equation

$$\Delta x^i \equiv x^i(\mathcal{Q}) - x^i(\mathcal{P}) = \Delta\lambda\, \dot{x}^i + \frac{1}{2}\,\Delta\lambda^2\, \ddot{x}^i + \mathcal{O}(\Delta\lambda^3) \tag{5.76}$$

and the product of their difference can be written

$$\Delta x^i\, \Delta x^j = \left(\Delta\lambda\, \dot{x}^i\right)\left(\Delta\lambda\, \dot{x}^j\right) + \mathcal{O}(\Delta\lambda^3), \tag{5.77}$$

both expressions being correct through second order in $\Delta\lambda$. Finally, using these in the expression (5.75) for $v^j(\mathcal{Q})$, we find

$$v^j(\mathcal{Q}) = \left[\delta^j_m - \Delta x^i\, \Gamma^j_{mi} + \frac{1}{2}\,\Delta x^k\, \Delta x^l \left(\Gamma^j_{nk}\, \Gamma^n_{ml} - \mathcal{G}^j_{mkl}\right)\right] v^m. \tag{5.78}$$

This last equation prescribes a method for obtaining the components of the parallel propagated vector at the point $\mathcal{Q}$ based on the components of $\vec{v}$ at $\mathcal{P}$. The result is discrete in the sense that the components at $\mathcal{Q}$ are correct through second order in $\Delta\lambda$. We can think of multiplying the matrix representation of the components at $\mathcal{P}$ by a discrete operator of the form,

$$\left[v^j\right]_{\mathcal{Q}} = \left[P^j_m\right] \left[v^m\right]_{\mathcal{P}}, \tag{5.79}$$

with the matrix $\left[P^j_m\right]$ defined with components given by equation (5.78)

$$P^j_m = \delta^j_m - \Delta x^i\, \Gamma^j_{mi} + \frac{1}{2}\,\Delta x^k\, \Delta x^l \left(\Gamma^j_{nk}\, \Gamma^n_{ml} - \mathcal{G}^j_{mkl}\right). \tag{5.80}$$

5.8 Conservation Laws

We have mentioned throughout this thesis that a representation of the conservation law (1.2) on a manifold takes the form (1.4). The geometry of $\mathscr{M}$ is responsible for the spatially varying flux function $\boldsymbol{f}^{(i)}(\vec{x}, \boldsymbol{q})$, and the source term $\boldsymbol{\psi}(\vec{x}, \boldsymbol{q})$. In this section we derive the form of both the flux and source term as functions of the metric tensor and its derivatives.

The conservation law (1.2) is in *geometric* form so that no reference to a basis is needed. The divergence operation has geometric meaning independent of any coordinate system. In order to generate a numerical solution, however, it is necessary to write (1.2) in a specific coordinate system. We consider a coordinate system $\mathcal{X}$, with coordinate functions x^k for $k \in \{1, \cdots, n_d\}$, that covers a subset $\mathscr{U} \subset \mathscr{M}$ on which we are interested in computing the solution to the conservation law (1.2). Associated with $\mathcal{X}$ is a coordinate basis $\{\vec{e}_{(k)}\}$ and its co-basis $\{\tilde{e}^{(k)}\}$.

In order to write out the form of (1.2) in this basis, we need to know the rank of the flux vector (or tensor, see Section 5.6). Recall that the divergence of a vector is a scalar, and the divergence of a second rank tensor is a vector. We must, therefore, consider both scalar and vector conservation laws.

5.8.1 Scalar conservation laws

Scalar conservation laws describe the time-evolution of a scalar quantity q in a volume $\mathscr{V}$, whose value changes only as a result of the flux through the volume boundary $\partial\mathscr{V}$. The flux for a scalar conservation law is a vector $\vec{f}(q)$, a type $\binom{1}{0}$ tensor. For example, the conservation of mass density ρ of a fluid has a flux defined by the momentum $\rho\vec{u}$. In general, the scalar version of the conservation law (1.2) has the form

$$q_{,t} + \vec{\nabla} \cdot \vec{f}(q) = 0. \tag{5.81}$$

Using the second equality in the divergence of a vector (5.67), we write the conservation law on the coordinate basis as

$$q_{,t} + f^k(q)_{,k} = -\Gamma^k_{jk}\, f^j(q), \tag{5.82}$$

where the source term that has appeared involves the Christoffel symbols and therefore, by equation (5.61), the derivatives of metric coefficients. Notice that two of the indices in the Christoffel symbol are contracted on. This contraction allows us to use equation (5.64) to write (5.82) as

$$q_{,t} + \frac{1}{\sqrt{h}} \left(\sqrt{h}\, f^k(q) \right)_{,k} = 0, \tag{5.83}$$

where $h = det([h_{ij}])$ and we have absorbed the Christoffel symbol into a spatially-varying flux function

$$\vec{f}(\vec{x}, q) = \sqrt{h(\vec{x})}\, \vec{f}(q). \tag{5.84}$$

This last equation defines the specific functional form of the spatially-varying flux we have been referring to in equation (1.4).

5.8.2 Vector conservation laws

Many systems for which conservation laws are written conserve not just scalars, but vectors too. An example is a fluid that flows according to the conservation of mass, momentum, and energy. Mass and energy are both scalars, but the momentum is a vector. Vector conservation laws have fluxes that are rank two tensors. The vector form of the conservation law (1.2) is written

$$\vec{\boldsymbol{q}}_{,t} + \vec{\nabla} \cdot \overleftrightarrow{T}(\vec{\boldsymbol{q}}) = 0. \tag{5.85}$$

Using the second equality in the divergence for a second rank tensor (5.69), and taking the j^{th} component of the resulting vector we have

$$q^j_{,t} + T^{kj}_{,k} = -\Gamma^k_{mk}\, T^{jm} - \Gamma^j_{km}\, T^{km}. \tag{5.86}$$

In this last equation we have two sets of Christoffel symbols, one of which has a contraction of two indices as in the scalar equation case. The other set of Christoffel symbols, however, are new. We can, once again, use equation (5.64) to absorb the contracted Christoffel symbol into a spatially-varying flux function of the form (5.84). We are then left with one set of Christoffel symbols contracted with the components of the flux tensor,

$$q^j_{,t} + \frac{1}{\sqrt{h}} \left(\sqrt{h}\, T^{kj} \right)_{,k} = -\Gamma^j_{mk}\, T^{mk}. \tag{5.87}$$

This equation defines the specific functional form of the geometrically induced source term from equation (1.4),

$$\boldsymbol{\psi}^j(\vec{x}, \boldsymbol{q}) = -\Gamma^j_{mk}\, T^{mk}(\boldsymbol{q}), \tag{5.88}$$

where, in the last equation, we have written the tangent-vector components of the source term $\boldsymbol{\psi}$.

5.8.3 Systems of conservation laws

Most problems of interest, however, do not simply involve a single scalar or vector equation. Rather, they involve a system of both scalar and vector conservation laws. We have already seen such a system in Chapters 3 and 4 where examples involving the conservation laws for an inviscid fluid in n_d-dimensional motion were presented. This system consists of $n_d + 2$ equations that are presented in detail in Appendix B. Two of these equations are scalar equations representing the conservation of both mass, ρ, and energy, e, of the fluid. The other n_d equations represent conservation of each component, s^i, of the vector conservation law for the momentum, $\vec{s}$.

In order to apply high-resolution numerical methods to such a system, it is helpful to write the system in the standard form (1.2). To this end, the state vector

$$\boldsymbol{q} = (\rho, s^j, e)^T, \tag{5.89}$$

is defined with $n_d + 2$ components and $j \in \{1, \cdots, n_d\}$. The autonomous flux in the i^{th} coordinate direction is computed by gathering the vector components along $\vec{e}_{(i)}$ of the flux function for each individual equation. For example, the flux vector for the density equation is $\rho\,\vec{u}$ (see Appendix B) so that the first component of $\boldsymbol{f}^{(i)}$ is $\rho\,\vec{u}(\tilde{e}^{(i)})$. Similarly, the flux for the energy equation is $(p+e)\,\vec{u}$ so that the n_d+2 component of $\boldsymbol{f}^{(i)}$ is $(p+e)\,\vec{u}(\tilde{e}^{(i)})$, where p is the fluid pressure. Finally, the $1+j$ component of $\boldsymbol{f}^{(i)}$ is computed from the momentum flux tensor, $\overleftrightarrow{T}$ and has the form $\overleftrightarrow{T}(\tilde{e}^{(i)}, \tilde{e}^{(j)})$. Gathering each of these components, the flux along $\vec{e}_{(i)}$ can be written

$$\boldsymbol{f}^{(i)} = \left(\rho\,\vec{u}^i, T^{ij}, (p+e)\,u^i\right)^T, \tag{5.90}$$

where, as before, $j \in \{1, \cdots, n_d\}$, so that if $n_d = 2$ then T^{ij} represents the two components T^{i1} and T^{i2}.

It was shown in equation (5.87) that the representation of this set of equations on $\mathscr{M}$ will induce a source term for all vector components of $\boldsymbol{q}$. This source term is represented by $\boldsymbol{\psi}$ and will have the form

$$\boldsymbol{\psi} = \left(0, -\Gamma^j_{kl}\,T^{kl}, 0\right), \tag{5.91}$$

where, as demonstrated in (5.83), there is no source term for scalar equations.

Using this state vector, flux vector, and source term, the system of conservation laws represented in a coordinate basis that covers a subset, $\mathscr{U}$ of a manifold $\mathscr{M}$ has the form

$$\boldsymbol{q}_{,t} + \frac{1}{\sqrt{h}}\left(\sqrt{h}\,\boldsymbol{f}^{(i)}\right)_{,i} = \boldsymbol{\psi}, \tag{5.92}$$

where there is an implicit sum on i over the spatial directions $\{1, \cdots, n_d\}$. This last equation is written in the standard form for most high-resolution finite volume methods. In the chapters that follow, details of the development and implementation of a new class of wave propagation algorithms that generate high-resolution numerical solutions to (5.92) are presented.