Numerical Modeling of Extracorporeal Shock Wave Therapy with Finite Volume Methods

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Outline

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ESWT Background

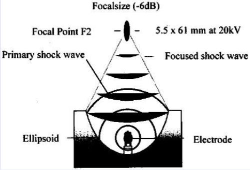
- Similar treatment to Shock wave Lithotripsy (ESWL)
- Shock waves are used to treat musculoskeletal conditions

Typical Treatment

- 1000-4000 shocks
- 1-4 shocks per second

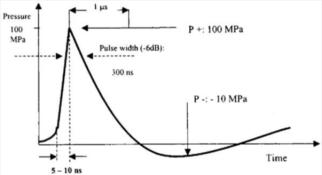
ESWT Shock Wave Properties

Shock Wave Generated by a spark plug source (electrohydraulic lithotripter)



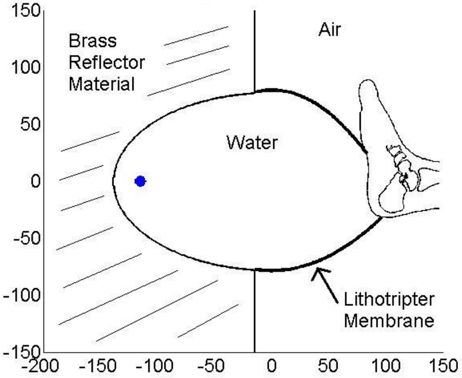
Typical Wave Form

Shock waves in lithotripsy are weak and can be modeled as discontinuities.



ESWT - material heterogeneities

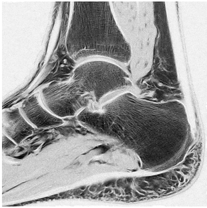
Numerical Representation of Lithotripter



Interfaces: Brass/Water, Air/Water, Water/Bone

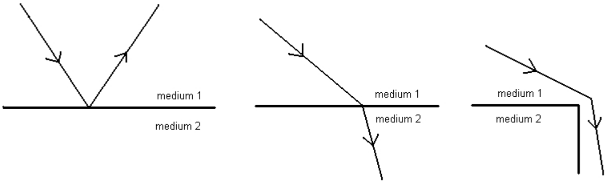
ESWT - material heterogeneities

Bone data is extracted from CT scans and modeled with uniform material properties.



ESWT Shock Wave Propagation

Wave Propagation in Heterogeneous media



Reflection Refraction Diffraction

Outline

Finite volume method

Finite volume methods

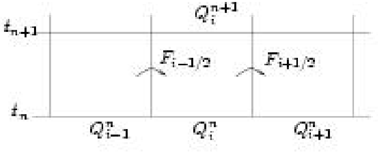
- Integral form of the conservation law,

$$\frac{\partial}{\partial t}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)\,dx = f(q(x_{i-1/2})) - f(q(x_{i+1/2}))$$

can be written in PDE form as $q_t + f(q)_x = 0$ and used to define the numerical method.

Finite volume method

Integral form:
$\frac{\partial}{\partial t} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)\,dx = f(q(x_{i-1/2})) - f(q(x_{i+1/2}))$



Define: $Q_i^n \approx \int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t_n)\,dx$

Numerical method: $Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2} - F_{i-1/2})$

Numerical flux: $F_{i-1/2} \approx \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(q(x_{i-1/2},t))\,dt$

The Riemann problem

The Riemann problem for $q_t + f(q)_x = 0$ has special initial data

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$

Solutions to this problem are used to define the numerical fluxes which update cell averages.

The Riemann Problem for Linear Acoustics

The Linear Acoustics Equations with spatially varying material parameters are:

$$\begin{bmatrix} p \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & K_0(x) \\ \frac{1}{\rho_0(x)} & 0 \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_x = 0,$$

This is a linear system $q_t + Aq_x = 0$. The eigenvalues and eigenvectors of A are :

$$\lambda^1 = -c_0(x),\ \lambda^2 = c_0(x),\ c_0 = \sqrt{K_0/\rho_0(x)}$$

$$r^1 = \begin{bmatrix} -Z_0(x) \\ 1 \end{bmatrix},\ r^2 = \begin{bmatrix} -Z_0(x) \\ 1 \end{bmatrix}$$

Riemann Problem for Linear Acoustics

General solution to linear acoustics problem:

$$\begin{bmatrix} p(x,t) \\ u(x,t) \end{bmatrix} = w^1(x - \lambda^1 t)r^1 + w^2(x - \lambda^2 t)r^2$$

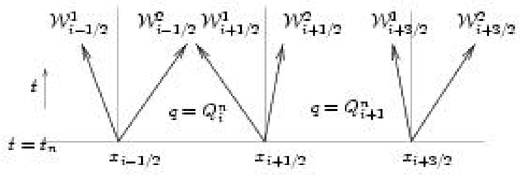
where w^1 and w^2 are dependent upon the initial condition and are found by setting t = 0 and solving

$$Rw = q_0$$

where R is the matrix of eigenvectors of A.

Riemann Problem for Linear Acoustics

Consider



We can represent the right and left states as linear combinations of the eigenvectors

$$q_r = w_r^1 r^1 + w_r^2 r^2 \text{ and } q_l = w_l^1 r^1 + w_l^2 r^2$$

The jump in q across the interface can be written as

$$(q_r - q_l) = \alpha^1 r^1 + \alpha^2 r^2 = \mathcal{W}^1 + \mathcal{W}^2$$

We can get alpha by solving $R\alpha = (q_r - q_l)$.
Combining these equations and solving for the middle state:

$$q^*(x,t) = ql + \alpha^1 r^1 = q_r - \alpha^2 r^2$$

REA Algorithm

- Reconstruct a piecewise polynomial function $\tilde{q}^n(, t_n)$ defined for all x, from the cell averages Q_i^n.
- Evolve the hyperbolic equation exactly (or approximately) with this initial data to obtain $\tilde{q}^n(x, t_{n+1})$ at the next time step. (Solve the Riemann problem at the cell interfaces!)
- Average this function over each grid cell to obtain new cell averages.

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Outline

Characteristics of ESWT

We would like our model for ESWT to:

- Capture wave behavior at sharp interfaces due to inhomogeneities
- Be able to represent weak shock wave as discontinuities
- Handle varying material parameters
- Model propagation of specific ESWT pressure wave form

Elasticity Equations

- There are two types of waves in a solid body:
  - P-waves are a result of compression or normal stresses
  - S-waves are a result of shearing
- Modeling these waves gives information about compression, tension and shear in the physical system.
- In water there are no shear waves and the linear elasticity equations are equivalent to acoustics.

IDEA: Model pressure wave in ESWT using elasticity equations.

Elasticity Equations

The linear elasticity equations are a result of assuming a linear relationship between the stress and strain (Hooke's law). The 3D Linear Elasticity Equations are:

$$
\begin{aligned}
\sigma_t^{11} - (\lambda + 2\mu)u_x - \lambda v_y - \lambda w_z &= 0 \\
\sigma_t^{22} - \lambda u_x - (\lambda + 2\mu)v_y - \lambda w_z &= 0 \\
\sigma_t^{33} - \lambda u_x - \lambda v_y - (\lambda + 2\mu)w_z &= 0 \\
\sigma_t^{12} - \mu(v_x + u_y) &= 0 \\
\sigma_t^{23} - \mu(v_z + w_y) &= 0 \\
\sigma_t^{31} - \mu(u_z + w_x) &= 0 \\
\rho u_t - \sigma_x^{11} - \sigma_y^{12} - \sigma_z^{13} &= 0 \\
\rho v_t - \sigma_x^{12} - \sigma_y^{22} - \sigma_z^{23} &= 0 \\
\rho w_t - \sigma_x^{13} - \sigma_y^{23} - \sigma_z^{33} &= 0
\end{aligned}
$$

Elasticity Equations - Linear System

This system can be written as

$$q_t + Aq_x + Bq_y + Cq_z = 0$$

where

$$q = \begin{bmatrix} \sigma^{11} \\ \sigma^{22} \\ \sigma^{33} \\ \sigma^{12} \\ \sigma^{23} \\ \sigma^{31} \\ u \\ v \\ w \end{bmatrix}, A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda+2\mu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu \\ -1/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 \end{bmatrix}$$

The eigendecomposition of these matrices is used to find solutions to the Riemann problem.

2D Linear Elasticity Results

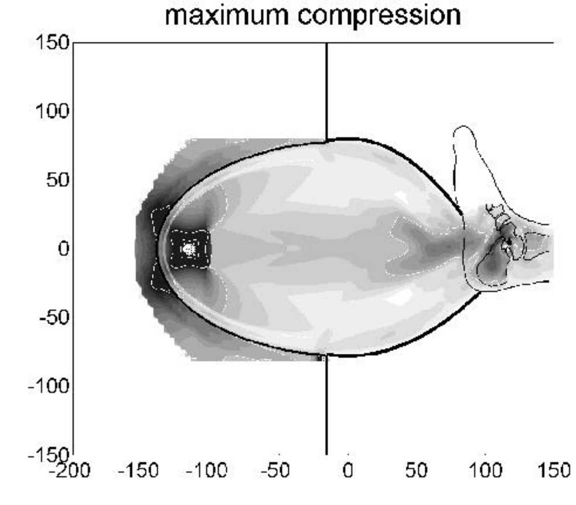
Click to start

2D Linear Elasticity Results

Click to start

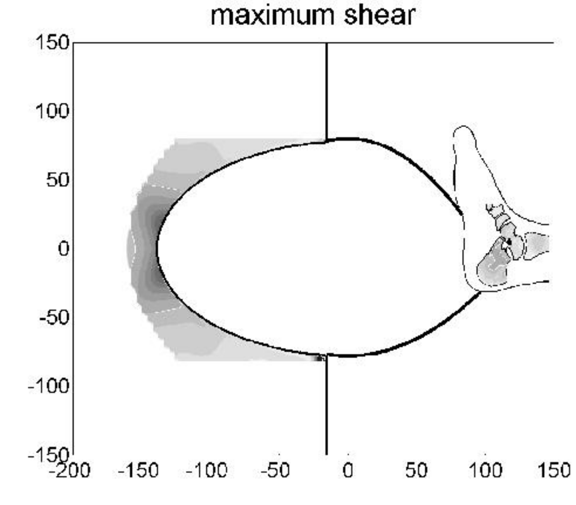
2D Linear Elasticity Results

Maximum compression



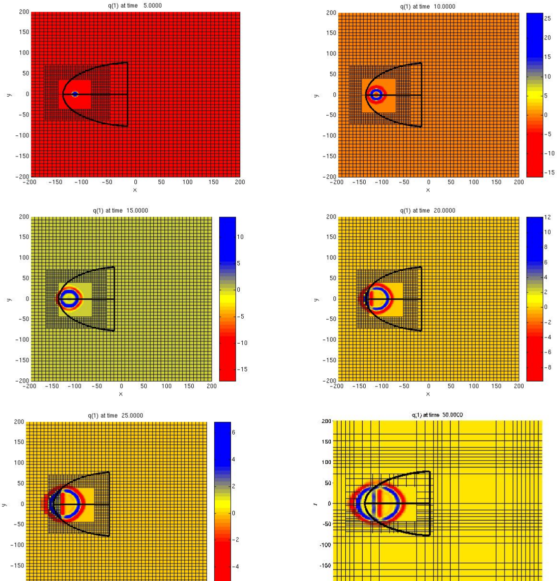
2D Linear Elasticity Results

Maximum shear



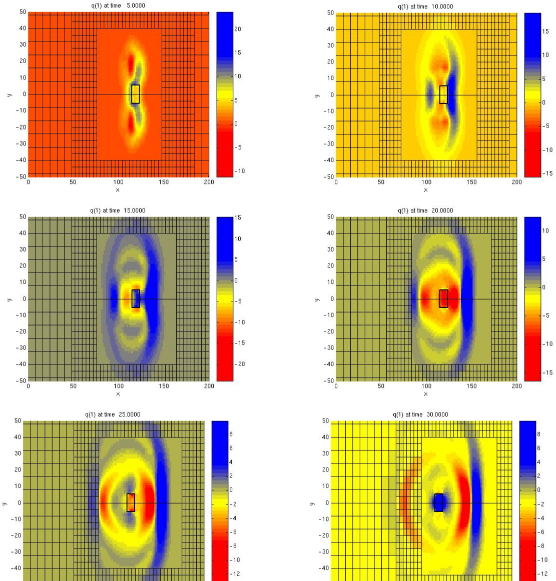
3D Linear Elasticity Results

Ellipsoidal reflection, focusing of pressure wave



3D Linear Elasticity Results

Pressure wave hitting a cylindrical inclusion:



Limitation of Linear Elasticity Equations

No nonlinear behavior can develop in this system - we can not model the ESWT pressure wave with this system.

Euler Equations of Gas Dynamics

- System of equations which models compressible, inviscid flow (Navier-Stokes with no viscosity).
- Nonlinear system of equations that is able to model the ESWT wave form when using the appropriate equation of state.

Euler Equations of Gas Dynamics

In conservative form the 3D Euler equations are:

$$q_t + f(q)_x + g(q)_y + h(q)_z = 0$$

where

$$q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{bmatrix}, \; f(q) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ (E+p)u \end{bmatrix},$$

$$g(q) = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v w \\ (E+p)v \end{bmatrix}, \; h(q) = \begin{bmatrix} \rho w \\ \rho u w \\ \rho v w \\ \rho w^2 + p \\ (E+p)w \end{bmatrix}$$

Here ρ is the density, p is the pressure, E is the total energy which we often decompose as $E = \rho e + \frac{1}{2}\rho u^2$, and e is the internal energy. These equations represent conservation of mass, momentum and energy.

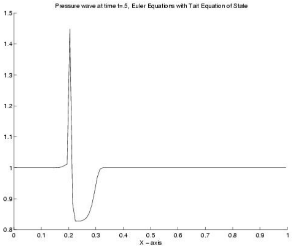
Tait equation of state (EOS)

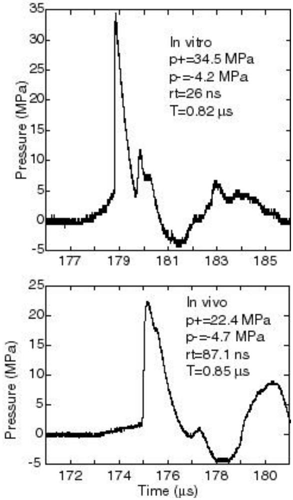
To close the above system we need a relationship between ρ, p and e (state variables). For ESWT we use:

$$\frac{p + p_\infty}{p_0 + p_\infty} = \left(\frac{\rho}{\rho_0}\right)^\gamma .$$

which is known as the Tait or stiffened gas EOS.

Pressure wave form





Limitation of Euler equations

Does not model solid dynamics, can not provide information about shear stresses.

Outline

Future Work

- A better model/integrated Riemann solver
- Stability issues?
- Correlation of results to experimental data
- Mapped grids for simple geometries
- Resolve memory issues
- Model shock wave behavior with 3D bone geometry