Some Traffic Flow Models Illustrating Interesting Hyperbolic Behavior

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Abstract. This note was prepared for a minisymposium on traffic flow at the SIAM Annual Meeting, July 10, 2001. It contains a brief discussion of two traffic flow models that illustrate some interesting aspects of the theory of nonlinear hyperbolic conservation laws. The first is a model of night-time driving that gives a nonconvex flux function in the classical LWR model, but for which the standard Oleinik entropy condition is not the correct admissibility condition. The second example includes a stiff source term and was designed to illustrate some basic features of detonation waves in a relatively simple context.

1 Introduction

This note concerns two different traffic flow models that were devised to illustrate some interesting aspects of hyperbolic theory for pedagogic purposes. The first is a model of night-time traffic that leads to a nonconvex flux function in the LWR model (the classical scalar conservation law developed by Lighthill & Whitham [8] and Richards [10]). However, the classical approach to selecting a weak solution based on the convex hull construction fails to produce the physically-correct solution in this case, due to the anisotropic behavior of traffic flow. This equation also exhibits an instability leading to measure-valued solutions.

The second example includes a stiff source term and was designed to illustrate some basic features of detonation waves in a relatively simple context.

In both examples $\rho(x,t)$ is used to denote the density of traffic in a continuum model, measured in units of "cars per car length" so that $0 \le \rho \le 1$. The LWR model consists of the scalar conservation law

$$\rho_t + f(\rho)_x = 0 \tag{1}$$

together with the flux function

$$f(\rho) = \rho U(\rho), \tag{2}$$

where $U(\rho)$ is a specified velocity function that is assumed to depend only on the density.

We will also use a car-following model in which the position $X_k(t)$ of individual cars (with $X_k < X_{k+1}$) is tracked via the system of ordinary differential equations

$$X_k'(t) = U(\rho_k(t)). \tag{3}$$

The local density $\rho_k(t)$ observed by the kth driver at time t is

$$\rho_k(t) = \frac{1}{X_{k+1}(t) - X_k(t)}. \tag{4}$$

This is the reciprocal of the "headway" $\Delta_k(t) = X_{k+1}(t) - X_k(t)$ seen by this driver.

2 Night-time traffic flow

Consider traffic traveling on an unfamiliar mountain road at night. In this situation it is often easier to drive quickly if there are other cars ahead on the road, since their tail lights indicate how the road

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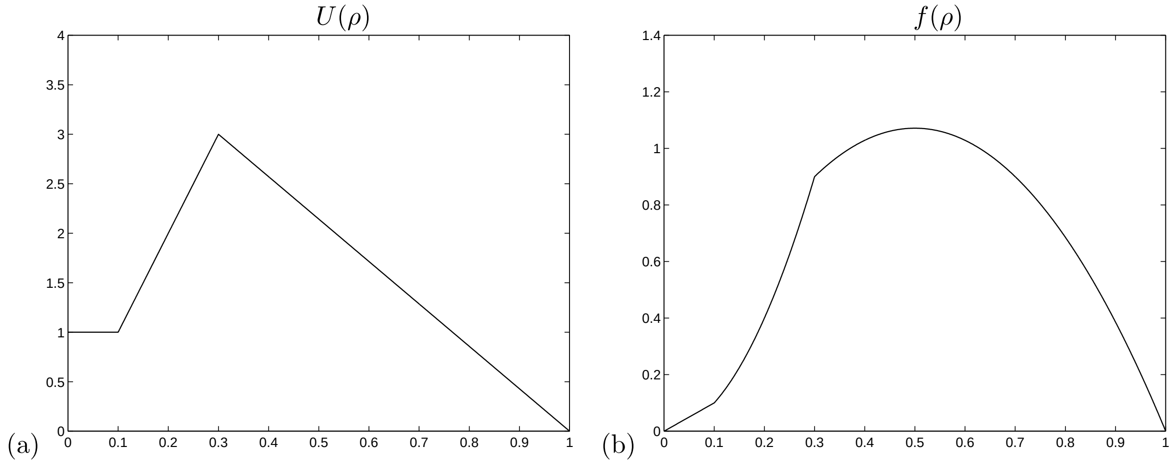

Figure 1: Traffic flow model for night-time driving. (a) The velocity function $U(\rho)$. (b) The resulting flux function $f(\rho)$.

twists and turns. When faced with empty road ahead, on the other hand, the driver's speed should be limited by the distance the headlights can illuminate. So a reasonable model for the velocity $U(\rho)$ as a function of density ρ might look something like Figure 1(a), with the velocity constant at some value U_0 for low density, then increasing for some range of density, and finally decreasing as in classical models if the density is sufficiently large. The specific function used here is given by

$$U(\rho) = \begin{cases} U_0 & \text{if } \rho < \rho_a \\ c\rho & \text{if } \rho_a \le \rho \le \rho_b \\ U_1(1-\rho) & \text{if } \rho > \rho_b \end{cases} \tag{5}$$

with $\rho_a = 0.1$, $\rho_b = 0.3$, $U_0 = 1$ and then

$$U_{\max} = \frac{\rho_b U_0}{\rho_a}, \quad c = \frac{U_{\max} - U_0}{\rho_b - \rho_a}, \quad U_1 = \frac{U_{\max}}{1-\rho_b}. \tag{6}$$

The corresponding flux function (2) for the standard LWR scalar model is illustrated in Figure 1(b). This flux function is not convex.

Consider the Riemann problem with $\rho_l = 1$ and $\rho_r = 0$, which corresponds to cars waiting for the light to turn green. If we apply the classical theory of nonconvex scalar conservation laws (*e.g.*, [7]) to this system, we would choose the weak solution consisting of a shock from $\rho_r = 0$ to the state ρ_b as shown in Figure 2(a), followed by a rarefaction wave to $\rho_l = 1$. This is the unique weak solution that satisfies the "Oleinik entropy condition", since

$$\frac{f(\rho) - f(\rho_b)}{\rho - \rho_b} \ge s \ge \frac{f(\rho) - f(\rho_r)}{\rho - \rho_r} \tag{7}$$

for all ρ between ρ_b and ρ_r.

However, this is **not** the solution that would be observed in "reality", *i.e.*, if the car-following model is used. Instead the lead car will drive at speed $U_0 < s^*$ and the cars following will therefore be constrained to this speed. The physically correct weak solution is the one shown in Figure 2(b). The density jumps from $\rho_r = 0$ to a large value $\hat{\rho}$ with the property that $U(\hat{\rho}) = U_0$. This is followed by a region where $\rho = \hat{\rho}$ (many cars all going at speed U_0) and finally a rarefaction wave to ρ_l. After the lead car (which sees density $\rho = 0$ ahead) takes off at speed U_0, the next car (which initially saw density $\rho = 1$ ahead) starts to accelerate through a rarefaction, with falling density. Once the density reaches the value $\hat{\rho}$, however, this car will stop accelerating and will simply follow at speed U_0. Each car in turn will have the same behavior.

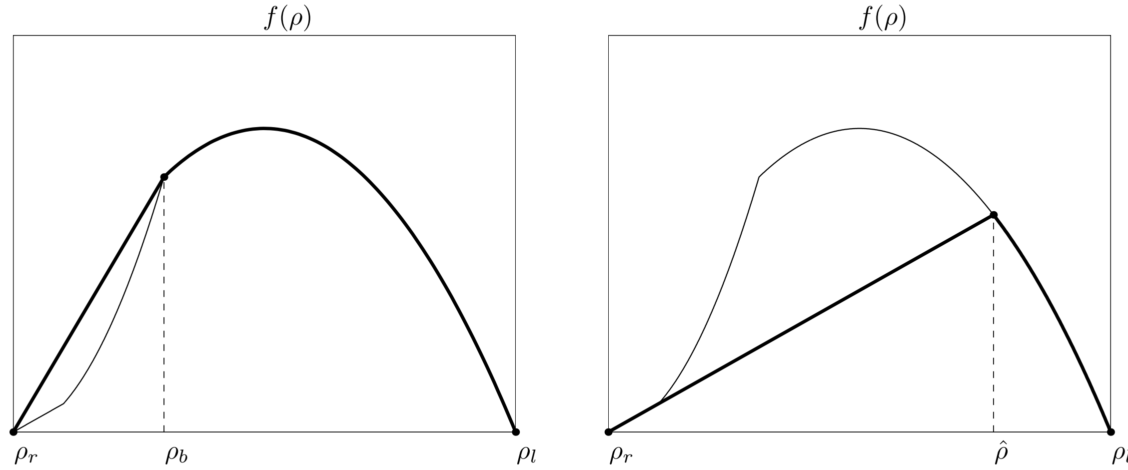

Figure 2: (a) The weak solution satisfying the Oleinik entropy condition. (b) The weak solution that agrees with the car-following model.

This is also a weak solution to the conservation law, but does not satisfy the standard admissibility condition given by Oleinik's entropy condition, since the shock speed $f(\hat{\rho})/\hat{\rho}$ is greater than the characteristic speed $f'(\hat{\rho})$. Oleinik's entropy condition is not the correct admissibility condition because it is based on the assumption that we wish to compute the vanishing-viscosity solution to the conservation law, which is obtained by adding a viscous term $\epsilon\rho_{xx}$ to the equation and letting $\epsilon \to 0$. This is not correct here because the car-following model is anisotropic. Unlike gas dynamics, where gas molecules respond to stimulus from all sides, we assume the driver of a car responds only to the distance to the car ahead and ignores the location of the car behind, and all other cars. See [1], [3], [11] for some other discussions of the anisotropic assumption in traffic flow.

It is possible to modify the car-following model to make it isotropic, in which case the vanishing-viscosity solution of Figure 2(a) will be observed. Simply replace the definition (4) of $\rho_k(t)$ by

$$\rho_k(t) = \frac{1}{2}\left(\frac{1}{X_{k+1}(t) - X_k(t)} + \frac{1}{X_k(t) - X_{k-1}(t)}\right) \tag{8}$$

so that the kth driver computes the density by averaging the local density ahead and behind. In this case the lead car can drive at a speed greater than U_0 if there is a car close behind. For the night-time mountain road example, however, this is not realistic and a different admissibility criterion is needed to select the correct weak solution. In particular, it appears that we need to require that the speed s of a shock connecting two states ρ_l and ρ_r satisfy

$$s \le U(\rho_r). \tag{9}$$

2.1 Instability and clustering

The night-time model exhibits another interesting feature. Suppose we take initial data consisting of a set of cars that are uniformly spaced (constant headway Δ^0 apart) on an otherwise empty road, so

$$\rho(x,0) = \begin{cases} \rho^0 = 1/\Delta^0 & \text{if } x_1 < x < x_2 \\ 0 & \text{otherwise }. \end{cases} \tag{10}$$

The resulting behavior depends on the value of ρ^0 relative to the structure of $U(\rho)$ in (5). This is illustrated in Figure 3 where the car-following model is solved for different values of Δ^0. If $\Delta^0 = 12$, then $\rho^0 = 1/12 < \rho_a$ and the cars are far enough apart that they all go at speed U_0. At the other extreme, if $\Delta^0 = 3$ then $\rho^0 = 1/3 > \rho_b$ and the road is sufficiently congested that a standard shock wave is observed, since $f(\rho)$ is convex for $\rho_b < \rho < 1$. For intermediate values of density $\rho_a < \rho < \rho_b$, however, the solution is more interesting, with the uniform traffic breaking up into platoons of cars that travel together at speed roughly U_0 with gaps in between. This is caused by the fact that uniformly spaced traffic with headway Δ^0 traveling at speed $U(1/\Delta^0)$ is unstable to small perturbations[2]. This is easy to see by considering the behavior of the first few cars. The lead driver sees $\rho = 0$ and drives at speed U_0. The second driver initially sees $\rho = \rho^0$ and so can drive faster. This causes the local density to decrease and hence the speed to increase until ultimately the second car falls into place a distance $1/\hat{\rho}$ behind the lead car, where $\hat{\rho}$ is the value shown in Figure 1 with $U(\hat{\rho}) = U_0$. But as the second car accelerates, the third driver observes a drop in density and hence this driver starts to *slow down*. This causes the fourth driver to go faster, and so on. The uniform flow naturally breaks up into platoons of two cars each. At later times this same behavior may repeat with pairs of 2-car platoons possibly merging into 4-car platoons and so on. This happens if the spacing between 2-car platoons gives them a "platoon density" that again falls in the unstable region. Note that if the original spacing is Δ^0 and the distance between the two cars in a 2-car platoon is $\hat{\Delta} = 1/\hat{\rho}$, then we expect the distance between platoons to be $2\Delta^0 - \hat{\Delta}$, from which we can compute the "platoon density".

The clustered solutions shown in Figure 3 can be interpreted as measure-valued solutions to the conservation law following the theory of DiPerna [4]. At each (x,t) the solution is not a single value $\rho(x,t)$ but rather a probability measure $\nu_{(x,t)}(\rho)$ on ρ. A classical solution corresponds to $\nu_{(x,t)}(\rho)$ being a delta function with strength $\rho(x,t)$. A stable flow consisting entirely of 2-car platoons with headway $\widehat{\Delta}$ between the two cars and headway $\bar{\Delta} > 1/\rho_a$ between the platoons (*i.e.*, platoon density $\bar{\rho} \equiv 1/\bar{\Delta} < \rho_a$) could be represented by

$$\nu_{(x,t)}(\rho) = \left(\frac{\widehat{\Delta}}{\widehat{\Delta} + \bar{\Delta}}\right)\delta(\rho - \hat{\rho}) + \left(\frac{\bar{\Delta}}{\widehat{\Delta} + \bar{\Delta}}\right)\delta(\rho - \bar{\rho}). \tag{11}$$

This has the following interpretation. On the macroscopic scale, where the individual platoons are not visible, the density at any point is equal to either $\bar{\rho}$ or $\hat{\rho}$ with probabilities given by the relative fraction of the road covered by inter- and intra-platoon gaps, respectively.

3 A traffic flow model exhibiting "detonation" waves

A detonation wave traveling through a combustible gas consists of a shock wave that raises the temperature of the gas above the ignition temperature, followed by a thin "reaction zone" in which the gas burns. This classic "ZND structure" (after Zeldovich, von Neumann, and Döring) is described in many sources, *e.g.*, [2], [5], [6]. Majda [9] introduced a model problem for the combustion system that consists of Burgers' equation, modeling the fluid dynamics part, along with an equation for a quantity Z which plays the role of the "unburnt fraction" of gas. When coupled together appropriately with a source term, some key features of a detonation wave are observed.

Here we will consider a slight modification of this model in which Burgers' equation is replaced by the LWR traffic flow model. The resulting system can be viewed as a model for traffic flow in a situation that makes at least some physical sense and may lead to a more intuitive understanding of the dynamics of detonation waves.

Consider the flow of cars along a one-lane road with the density denoted by $\rho(x,t)$. Suppose that in addition there are cars parked along the side of the road, with density β, that wish to merge into the traffic. Let $Z(x,t)$ represent the fraction that are still alongside the road, and suppose they merge at some rate K. Then the "unmerged fraction" satisfies the equation

$$Z_t = -KZ, \tag{12}$$

while the conservation law for the density $\rho(x,t)$ gains a source term,

$$\rho_t + f(\rho)_x = K\beta Z. \tag{13}$$

[2]The fact that the model should exhibit this instability was pointed out to the author by Phillipe LeFloch

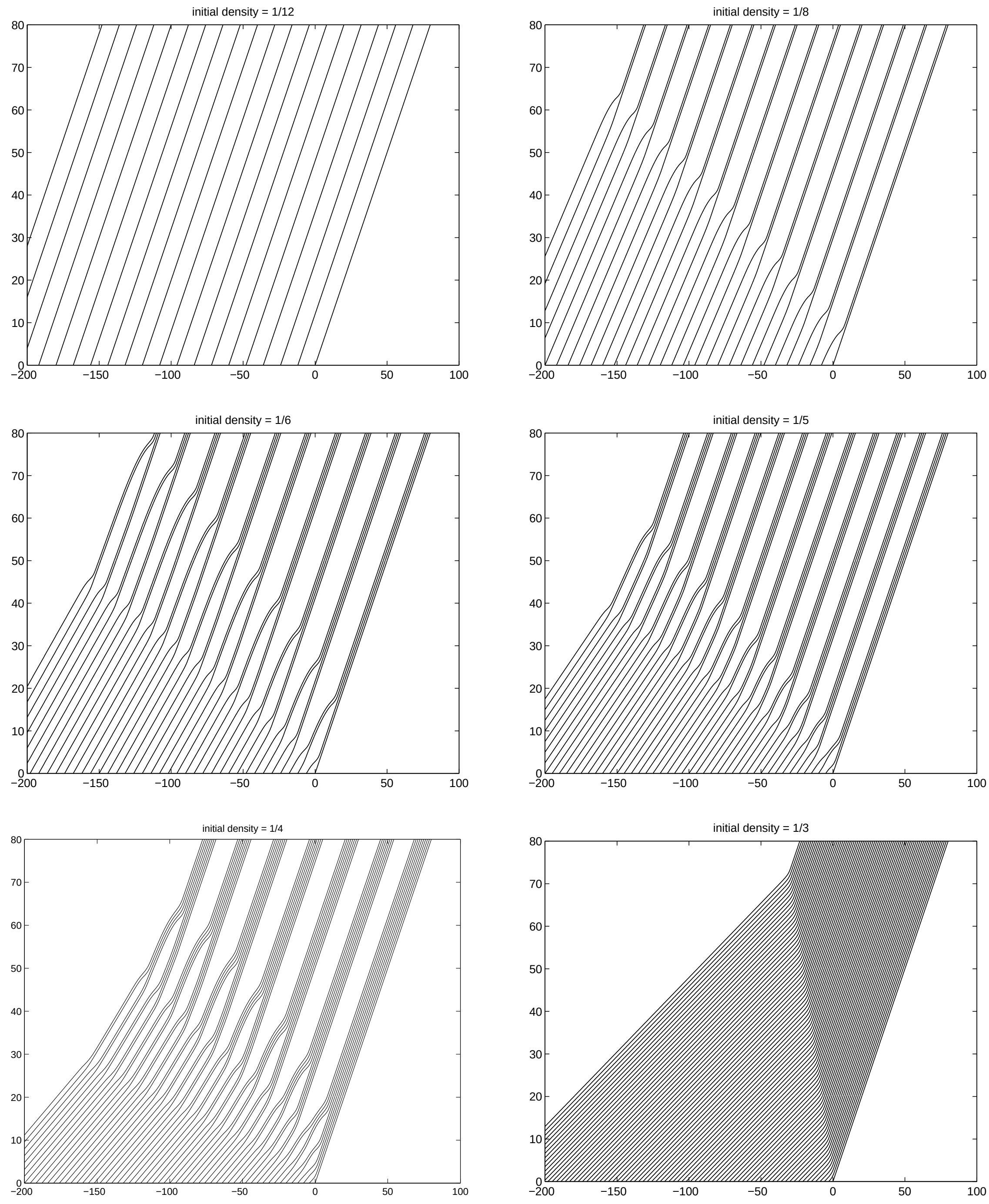

Figure 3: Vehicle trajectories in the x-t plane computed using the car-following model for the night-time traffic model with various initial densities.

For the traffic flux we use the classic convex function

$$f(\rho) = \rho(1-\rho) \tag{14}$$

resulting from $U(\rho) = 1-\rho$. The model (12), (13) is analogous to a simple model for combustion in which the Euler equations of gas dynamics are coupled with an equation of the form (12) for the unburnt fraction of gas, and the conservation of energy equation gains a source term similar to (13) corresponding to the heat release of combustion converting chemical energy into internal energy.

If K were a constant, the system (12), (13) would not be very interesting. All the cars would merge into the traffic simultaneously and the road would simply become more congested everywhere.

Instead, suppose the drivers in the parked cars are cautious and will not merge unless the speed of cars on the road is sufficiently low. Since velocity decreases as density increases, this means they will not merge unless ρ is sufficiently large already. This suggests that

$$K(\rho) = \begin{cases} 0 & \text{if } \rho < \rho_{\mathrm{I}} \\ K_1 & \text{if } \rho > \rho_{\mathrm{I}}, \end{cases} \tag{15}$$

for some value ρ_{I}, where I stands for "ignition". This is analogous to the situation with detonation waves, since the chemical reactions of combustion are exothermic and release heat, but only take place if the temperature is already sufficiently high.

As in combustion, it now is possible to observe a "detonation wave". Suppose the initial density is below ρ_{I} but a traffic-jam shock moves up the road that raises the density above ρ_{I}. Then the shock will be followed by a "reaction zone" (thin, if K is large) in which all available cars merge. Figure 4 shows the development and propagation of such a wave for initial Riemann data

$$\begin{aligned} \rho_l &= 0.60, \qquad & \rho_r &= 0.85, \\ Z_l &= 1, \qquad & Z_r &= 0, \end{aligned} \tag{16}$$

and the parameter values $\rho_{\mathrm{I}} = 0.65$, $\beta = 0.05$, $K = 3$. This is very similar to the ZND structure observed in a detonation wave. The shock wave raises the density to some value ρ_{vN} greater than ρ_{I} and then the density falls through the reaction zone. Here "vN" stands for von Neumann, since the corresponding state just behind the shock wave in a detonation is called the von Neumann state.

An apparent paradox is the fact that the density ρ observed by a driver *falls* as the car moves through the reaction zone, even though additional cars are joining the traffic. We might expect the density to rise even higher behind the shock through this zone. But a rising density would lead to a compression wave that would travel faster than the shock, and must then merge into the shock. This suggests that we cannot have a traveling wave with that structure. The falling density observed corresponds to accelerating traffic. Cars move out of the reaction zone fast enough that the spreading of cars due to this acceleration is not completely offset by the addition of new cars from the source term. This is analogous to the fact that the pressure falls through the reaction zone of a detonation wave in spite of the exothermic chemical reactions taking place, because the gas is also rapidly expanding.

As $K_1 \to \infty$ the width of the reaction zone shrinks, and we might think of representing the processes in the reaction zone by a delta-function source term concentrated at a point st moving with some constant velocity s, leading to the model

$$\rho_t + f(\rho)_x = d\delta(x - st). \tag{17}$$

The source strength d is related to β and also to the speed s. The total mass of cars entering the road over time t is βst since all cars parked along a section of road of length st have merged in this time. Hence the source density per unit time must be

$$d = \beta s.$$

The equation (17) has a solution that consists of a discontinuity propagating at speed s provided that a generalized Rankine-Hugoniot relation holds that takes into account the singular source term,

$$s(\rho_l - \rho_r) = f(\rho_l) - f(\rho_r) + \beta s. \tag{18}$$

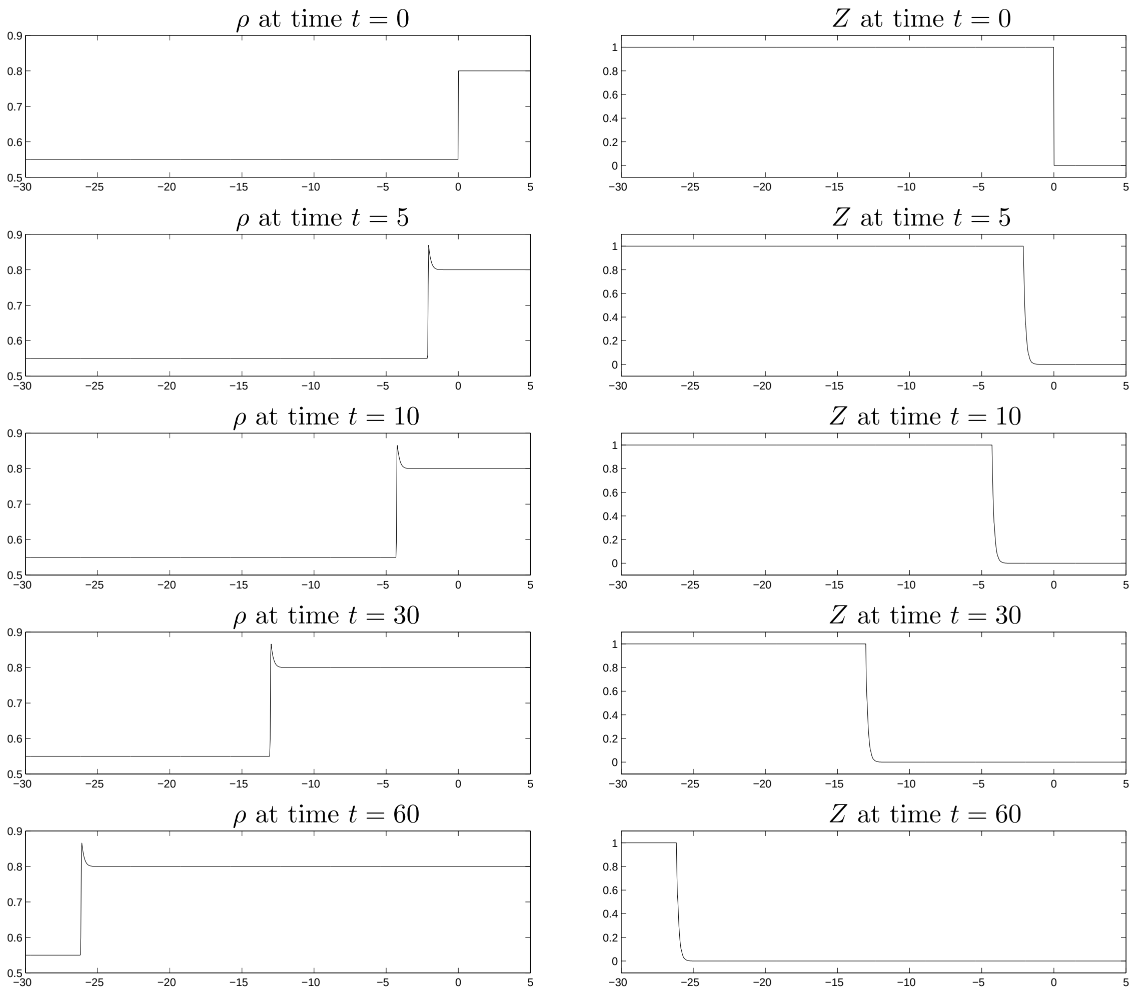

Figure 4: A traffic-flow detonation wave, exhibiting the classic ZND structure of a shock followed by a reaction zone.

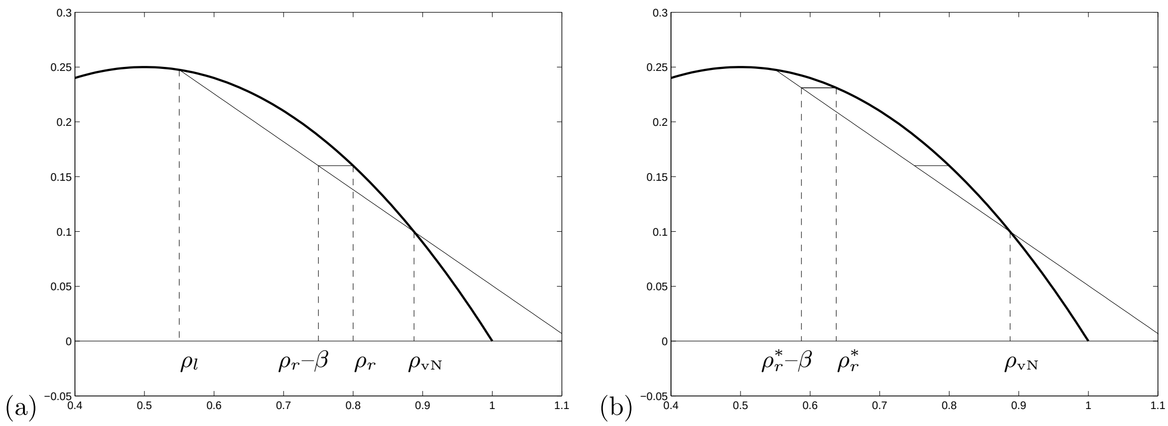

Figure 5: The flux function $f(\rho)$ (heavy line) and the line with slope s equal to the speed of a detonation wave with heat release $\beta = 0.05$. (a) Strong detonation connecting ρ_l to ρ_r. A shock from ρ_l to $\rho_{\rm vN}$ is followed by the reaction zone. (b) A weak detonation with the same speed s connects ρ_l to ρ_r^* but is dynamically unstable.

From this we can determine that

$$s = \frac{f(\rho_r) - f(\rho_l)}{(\rho_r - \beta) - \rho_l}. \tag{19}$$

Note that the ZND structure is not captured in this model — the shock and reaction zone are compressed into a single discontinuity connecting ρ_l directly to ρ_r. The corresponding analysis in combustion is called the **Chapman-Jouget theory**. The density peak $\rho_{\rm vN}$ is lost from the resulting solution, but conservation of mass (taking into account both types of cars) gives the correct speed s. The value $\rho_{\rm vN}$ can then be recovered from ρ_l and s since we know that $\rho_{\rm vN}$ and ρ_l are connected by an ordinary shock with speed $s = 1 - \rho_l - \rho_{\rm vN}$ from the standard Rankine-Hugoniot condition.

Figure 5(a) illustrates the jump condition (19) geometrically for the flux function (14). The detonation speed s is the slope of the line from the left state $(\rho_l, f(\rho_l))$ to the point $(\rho_r - \beta, f(\rho_r))$. This latter point is offset from the curve $f(\rho)$ by distance β. The point where this line again intersects the curve $f(\rho)$ is the von Neumann density $\rho_{\rm vN}$ that occurs just behind the shock wave in the ZND structure of Figure 4. This follows since this shock must be propagating at the same speed s as the idealized detonation wave. In the ZND structure, the shock from ρ_l to $\rho_{\rm vN}$ is followed by the reaction zone in which the density falls from $\rho_{\rm vN}$ to ρ_r. Note that for the homogeneous conservation law with data $\rho_{\rm vN}$ and ρ_r we would obtain a rarefaction wave moving with characteristic velocity $f'(\rho)$ at each point in between. However, from Figure 5(a) we see that $f'(\rho)$ is more negative than s at each of these points, so this rarefaction wave structure would disappear into the detonation shock. While it is true that cars accelerate and spread out through the reaction zone, it is qualitatively different from a rarefaction wave. Note in particular that the reaction zone has fixed width as time advances, and so the solution of Figure 4 approaches a traveling wave, whereas a rarefaction wave would continue to expand self-similarly as time increases.

3.1 Weak and strong detonations

Figure 5(b) illustrates another interesting feature of detonation waves. For this same left state ρ_l and speed s, there appears to be another right state ρ_r^* that could also be connected to ρ_l by a detonation wave propagating at speed s. This point ρ_r^* is located at the second point where the curve $f(\rho)$ is horizontal distance β to the right of the line with slope s through $f(\rho_l)$. This is called the **weak detonation** associated with this (ρ_l, s), whereas the structure illustrated in Figure 5(a) is called the **strong detonation** (since $\rho_r - \rho_l > \rho_r^* - \rho_l$).

However, if we numerically solve the equations (12) and (13) with initial data (ρ_l, ρ_r^*), we obtain the solution shown in Figure 6. This is qualitatively different from the detonation wave of Figure 4. There is still a detonation wave and a reaction zone, but these are now followed by an ordinary rarefaction wave in which ρ continues to decrease even though Z is essentially zero. This occurs because our previous argument about rarefaction waves disappearing into the detonation shock fails for weak detonations. Near ρ_r^* in Figure 5(b) we see that $f'(\rho)$ is less negative than s, so a rarefaction wave can escape from the detonation structure. This means that the structure predicted by the Chapman-Jouget analysis is not correct, since the derivation of s and ρ_{vN} was based on the assumption that all the action occurs in this reaction zone, and indeed the detonation wave in Figure 6 does *not* have the same speed or peak density as in Figure 4. In the next section we determine the solution seen in Figure 6 analytically.

3.2 Chapman-Jouget detonations

Figure 7 illustrates the procedure that must be used to derive the correct solution to the problem just described with initial data (ρ_l, ρ_r^*). The state ρ_l cannot be connected directly to ρ_r^* by a detonation wave even though the jump conditions are satisfied, since this weak detonation structure is dynamically unstable. Instead a rarefaction wave appears behind the detonation wave that can spread out away from it since the characteristic velocity is greater than the speed of the detonation wave. As a result the density just behind the reaction zone is greater than ρ_r^*. Increasing the right-state value causes a decrease in the speed of the detonation; the straight line with slope s is pivoted upwards through ρ_l as indicated in Figure 7 until it reaches a slope s_{CJ} corresponding to a state ρ_{CJ} as indicated in Figure 7. For this particular right state the characteristic speed $f'(\rho_{\text{CJ}})$ is equal to the detonation speed s_{CJ}. This determines the state ρ_{CJ} by solving the equation

$$f'(\rho_{\text{CJ}}) = \frac{f(\rho_{\text{CJ}}) - f(\rho_l)}{(\rho_{\text{CJ}} - \beta) - \rho_l}.$$

The solution observed numerically in Figure 6 and illustrated in Figure 7 thus consists of:

- a shock from ρ_l to ρ_{vN}^*, the point indicated in Figure 7 where the line with slope s_{CJ} intersects $f(\rho)$ again,
- a reaction zone from ρ_{vN}^* to ρ_{CJ},
- a rarefaction wave from ρ_{CJ} to ρ_r^*. The left edge of this rarefaction has speed $f'(\rho_{\text{CJ}})$, equal to the speed of the detonation wave, so that it remains attached to the detonation wave.

We can view the state ρ_{CJ} as the one corresponding to the weakest possible strong detonation that could develop from the given data. Also note that for fixed ρ_l and β, the speed s_{CJ} is the slowest speed (*i.e.*, least negative) that any detonation wave can have, and that at this speed there is only a single detonation possible. At lower speeds there would no longer be a point on the curve $f(\rho)$ a distance β to the right of the line. As the speed is increased (pivoting the line downwards through ρ_l), this solution splits into two possibilities, the weak and strong detonations that exist for greater speeds. For given ρ_l and β this speed s_{CJ} is called the **Chapman-Jouget speed**.

An analogous theory holds for real detonation waves in combustible gas dynamics. Weak detonations are seen only in very special circumstances, and generally only strong detonations and Chapman-Jouget detonations are dynamically stable. Chapman-Jouget detonations are often observed in practice since weak detonations tend to evolve into these.

This simple traffic model illustrates some features of real detonation waves and the notation has been borrowed from that theory, but the analogy is not certainly not perfect and the situation in gas dynamics is still richer.

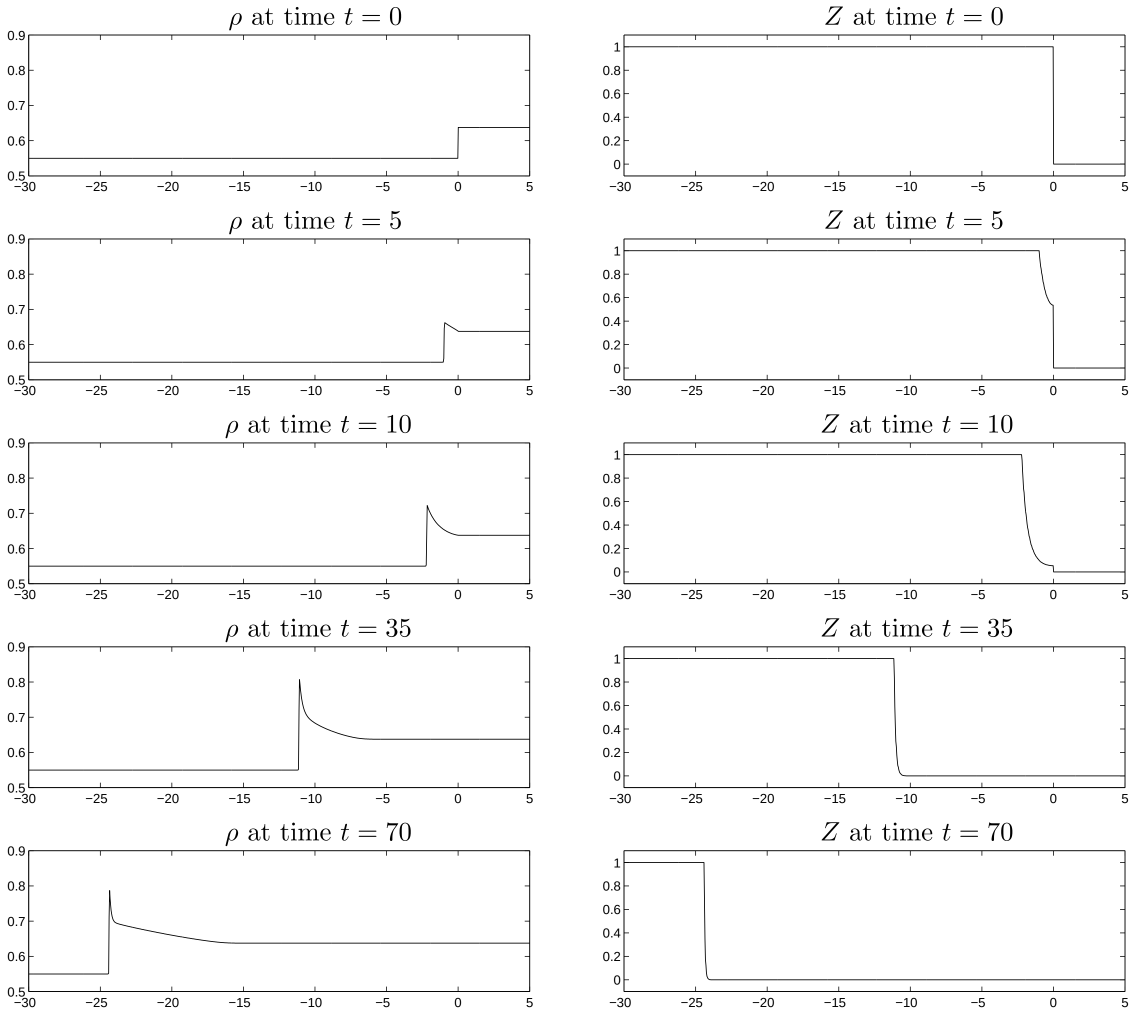

Figure 6: Numerical solution using the data (ρ_l, ρ_r^*) corresponding to a weak detonation wave. The solution evolves into a CJ detonation followed by a rarefaction wave. Illustrated for $\rho_I = 0.65, \beta = 0.05,\ K = 0.05$

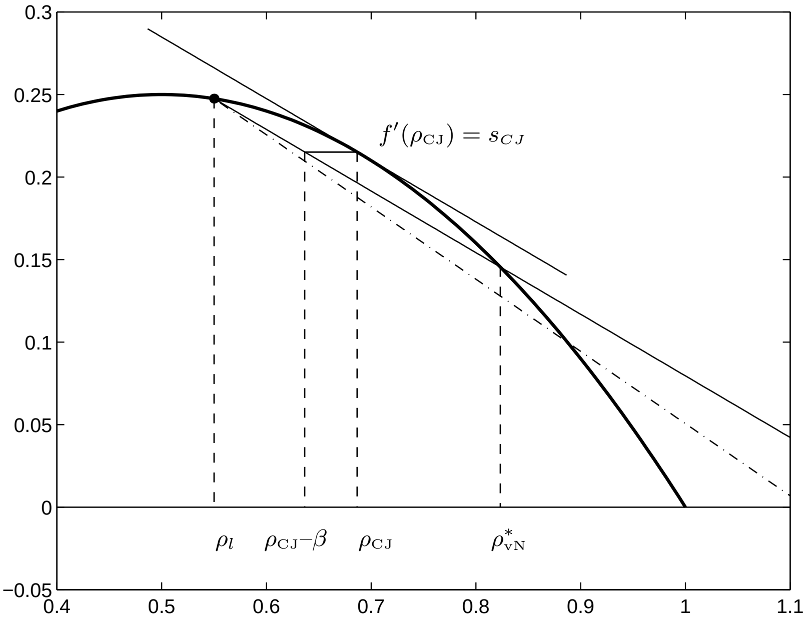

Figure 7: The flux function $f(\rho)$ (heavy line) and the line with slope s_{CJ} equal to the speed of a Chapman-Jouget detonation wave for this particular value of ρ_l and β. This line is pivoted up from the line shown in Figure 5 (shown here as a -.-. line). A shock from ρ_l to $\rho^*_{\rm vN}$ is followed by the reaction zone to $\rho_{\rm CJ}$. The tangent to $f(\rho)$ at $\rho_{\rm CJ}$ is also shown, illustrating that the slope at this point is equal to s_{CJ}.

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