Tsunami Modeling[1]

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1 Introduction

Appreciation for the danger of tsunamis among the general public has soared since the Indian Ocean tsunami of 26 December 2004 killed more than 200,000 people. Several other large tsunamis have occurred since then, including the devastating 11 March 2011 Great Tohoku tsunami generated off the coast of Japan. The international community of tsunami scientists has also grown considerably since 2004 and an increasing number of applied mathematicians have contributed to the development of better models and computational tools for the study of tsunamis. In addition to its importance in scientific studies and public safety, tsunami modeling also provides an excellent case study to illustrate a variety of techniques from applied and computational mathematics. This article combines a brief overview of tsunami science and hazard mitigation with descriptions of some of these mathematical techniques, including an indication of some challenging problems of ongoing research.

The term "tsunami" is generally used to refer to any large scale anomolous motion of water that propagates as a wave in a sizable body of water. Tsunamis differ from familiar surface waves in several ways. Typically the fluid motion is not confined to a thin layer of water near the surface as it is in wind-generated waves. Also the wavelength of the waves is much longer, sometimes hundreds of kilometers. This is orders of magnitude larger than the depth of the ocean (which is about 4 km on average) and so tsunamis are also sometimes referred to as "long waves" in the scientific literature. In the past, tsunamis were often called "tidal waves" in English because they share some characteristics with tides, which are the visible effect of very long waves propagating around the earth. However, tsunamis have nothing to do with the gravitational (tidal) forcing that drives the tides, and so this term is misleading and is no longer used. The Japanese word "tsunami" means "harbor wave", apparently because sailors would sometimes return home to find their harbor destroyed by mysterious waves they did not observe while at sea. Strong currents and vortices in harbors often cause extensive damage to ships and infrastructure even when there is no onshore inundation. Although the worst effects of a tsunami are often observed in harbors, the effects can be devastating in any coastal region. Because tsunamis have such a long wavelength, they frequently appear onshore as a flood that can continue flowing inward for tens of minutes or even hours before flowing back out. The flow velocities can also be quite large, with the consequence that even a tsunami wave with an amplitude of less than 1 meter can sweep people off their feet and do considerable damage to structures. Tsunamis arising from large earthquakes often result in flow depths greater than 1 meter, particularly along the coast closest to the earthquake, where runup can reach 10s of meters.

Tsunamis are generated whenever a large mass of water is rapidly displaced, either by the motion of the seafloor due to an earthquake or submarine landslide, or when a solid mass enters the water from a landslide, volcanic flow, or asteroid impact. The largest tsunamis in recent history, such as the 2004 or 2011 events mentioned above, were all generated by megathrust subduction zone earthquakes at the boundary of oceanic and continental plates. Offshore from many continents there is a subduction zone where plates are converging. The denser material in the oceanic plate subducts beneath the lighter continental crust. Rather than sliding smoothly, stress builds up at the interface and is periodically released when one plate suddenly slips several meters past the other, causing an earthquake during which the seafloor is lifted up in some regions and depressed in others. All of the water above the seafloor is lifted or falls along with it, creating a disturbance on the sea surface that propagates away in the form of waves. See Figure 1 for an illustration of tsunami generation and Figure 2 for a numerical simulation of waves generated by the 2011 Tohoku earthquake off the coast of Japan.

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This article primarily concerns tsunamis caused by subduction zone earthquakes since they are a major concern in risk management and have been widely studied.

2 Mathematical models and equations of motion

Tsunamis are modeled by solving systems of partial differential equations (PDEs) arising from the theory of fluid dynamics. The motion of water can be very well modeled by the Navier-Stokes equations for an incompressible viscous fluid. However, these are rarely used directly in tsunami modeling since they would have to be solved in a time-varying three-dimensional domain, bounded by a free surface at the top and by moving boundaries at the edges of the ocean as the wave inundates or retreats at the shoreline. Fortunately, for most tsunamis it is possible to use "depth-averaged" systems of PDEs, obtained by integrating in the vertical z direction to obtain equations in two space dimensions (plus time). In these formulations, the depth of the fluid at each point is modeled by a function $h(x, y, t)$ that varies with location and time. The velocity of the fluid is described by two functions $u(x, y, t)$ and $v(x, y, t)$ that represent depth averaged values of the velocity in the x- and y-directions respectively. In addition to a reduction from three to two space dimensions, this eliminates the free surface boundary in z; the location of the sea surface is now determined directly from the depth $h(x, y, t)$. These equations are solved in a time-varying two-dimensional (x, y) domain since the moving boundaries at the shoreline must still be dealt with.

A variety of depth-averaged equations can be derived, depending on the assumptions made about the flow. For large-scale tsunamis, the so-called "shallow water" equations (also called the St. Venant or long-wave equations) are frequently used and have been shown to be very accurate. The assumption with these equations is that the fluid depth is sufficiently shallow relative to the wavelength of the wave being studied. This is generally true for tsunamis generated by earthquakes, where the wavelength is typically 10–100 times greater than the ocean depth.

The two-dimensional shallow water equations have the form

$$\begin{aligned} h_t + (hu)_x + (hv)_y &= 0, \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y &= -ghB_x, \\ (hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y &= -ghB_y, \end{aligned} \tag{1}$$

where subscripts denote partial derivatives, e.g. $h_t = \partial h/\partial t$. In addition to the variables h, u, v already introduced, which are each functions of (x, y, t), these equations involve the gravitational force g and the topography or seafloor bathymetry (underwater topography) denoted by $B(x, y)$. Typically $B = 0$ represents sea level while $B > 0$ is onshore topography and $B < 0$ represents seafloor bathymetry. Water is present wherever $h > 0$ and $\eta(x, y, t) = h(x, y, t)+B(x, y)$ is the elevation of the water surface. See Figure 3 for a diagram in one space dimension. During an earthquake B should also be a function of t in the region where the seafloor is deforming. In practice it is often sufficient to include this deformation in $B(x, y)$ while the initial conditions for the depth $h(x, y, 0)$ are based on the undeformed topography. The seafloor deformation then appears instantaneously in the initial surface $\eta(x, y, 0)$, which initializes the tsunami. In the remainder of this article, the term *topography* will be used for both $B > 0$ and $B < 0$ for simplicity.

If $B(x, y) < 0$ is constant (a flat bottom) then the "source terms" on the right hand side of these equations drop out and the equations model the conservation of mass (h) and momentum (hu, hv). Over a varying bottom, mass is still conserved but momentum is affected by the terrain, as seen for example in the reflection of waves at a shoreline and partial reflection when a wave interacts with underwater features. The term $\frac{1}{2}gh^2$ appearing in the momentum equations is the depth averaged "hydrostatic pressure" in a column of water of depth h. (This and all other terms in (1) should in fact also involve the fluid density ρ, but this cancels out everywhere if the density is assumed to be constant.)

The equations (1) are a nonlinear system of equations of hyperbolic type. Hyperbolic PDEs

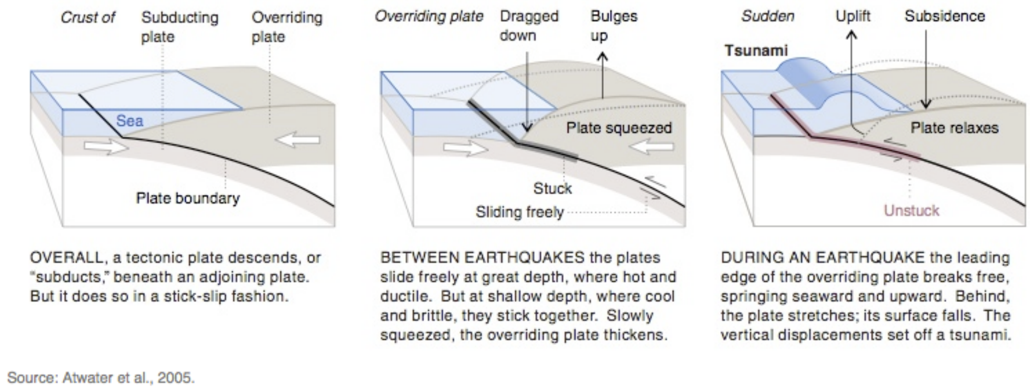

Figure 1: Illustration of the generation of a tsunami by a subduction zone earthquake.

frequently arise when waves are modeled mathematically, as described in [article on hyperbolic PDEs]. The amplitude of a tsunami in the deep ocean is generally very small relative to the water depth; typically less than a meter even for a large megathrust tsunami. Away from the coast these equations could be approximated by linearized equations with variable coefficients arising from the varying topography. Near shore, however, the amplitude of the wave is large relative to the depth of the fluid and the full nonlinear equations must be used to accurately model the interaction of a tsunami with the nearshore topography and the onshore inundation that occurs. Solutions to nonlinear hyperbolic PDEs can become discontinuous if a shock develops. In the case of the shallow water equations, a shock is also called a "hydraulic jump" and is a mathematical idealization of a thin region in which the depth and velocity of the fluid jumps from one value to another. Such regions frequently appear as a turbulent wave front (sometimes called a "turbulent bore") once the tsunami moves into sufficiently shallow water. The shallow water equations do not model this turbulent zone directly, but are frequently adequate to capture important quantities such as the depth and fluid velocities behind the bore and its propagation speed.

3 Uses of tsunami modeling

The PDEs describing a tsunami cannot be solved exactly, in general, and so numerical methods must be used to simulate the propagation and inundation of a tsunami. A brief description of how this might be done and some of the challenges that arise is given in Section 4, but first we motivate the need for numerical models by describing some common uses of such models.

3.1 Real-time warning systems

One natural use of a numerical model is to assist in issuing warnings in real time as a tsunami propagates across the ocean, and to determine what coastal regions should be evacuated. There are many challenges to doing this quickly and accurately. Accurate assessment is critical not only to insure that areas at risk are properly warned but also to avoid triggering evacuation in areas where it is not necessary, which can itself cause loss of life, serious financial impact, and decreased attention to future warnings. For a subduction zone megathrust earthquake it is often impossible to issue tsunami warnings quickly enough for areas along the nearby coastline. The tsunami may arrive in less than an hour, often sooner, and it is critical that residents understand the need to move to high ground when a major earthquake occurs. On the other hand, across the ocean the earthquake itself is not felt and so provides no direct indication of an impending tsunami, but there are several hours available to perform simulations and issue warnings.

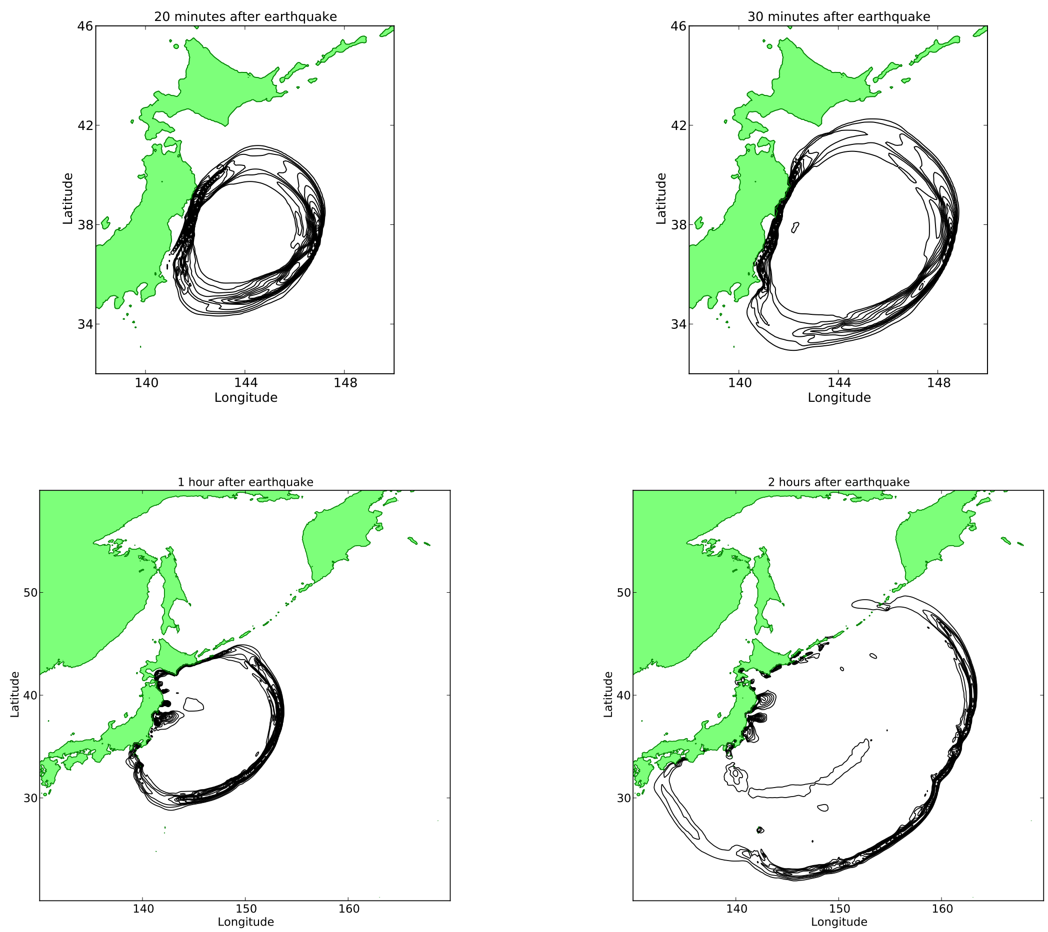

Figure 2: Propagation of the tsunami arising from the 11 March 2011 Tohoku earthquake off the coast of Japan at four different times from 20 minutes to 2 hours after the earthquake. Waves propagate away from the source region with a velocity that varies with the local depth of the ocean. Contour lines show sea surface elevation above sea level, in increments of 20 cm (top, at early times) and 10 cm (bottom, at later times). There is a wave trough behind the leading wave peak shown here, but for clarity the contours of elevation below sea level are not shown.

3.2 Tsunami source inversion

To perform tsunami simulations, it is necessary to estimate the source, i.e., the deformation of the sea floor that generates the tsunami, since this determines the initial conditions that are used to numerically solve the PDEs modeling tsunami propagation and inundation. There is generally no way to measure this directly, and so some form of *inverse problem* [pointer to article on inverse problems?] must be solved to obtain an estimate of the deformation based on measurements that can be made, such as the earth motion due to seismic waves caused by the earthquake, or measurements of the tsunami itself. Initial estimates of the location and magnitude of an earthquake generally come from analyzing recordings of seismic waves, which are compression and shear waves that travel through the earth with much higher velocity than tsunamis and that are routinely recorded at hundreds of seismometers widely scattered around the world. From the

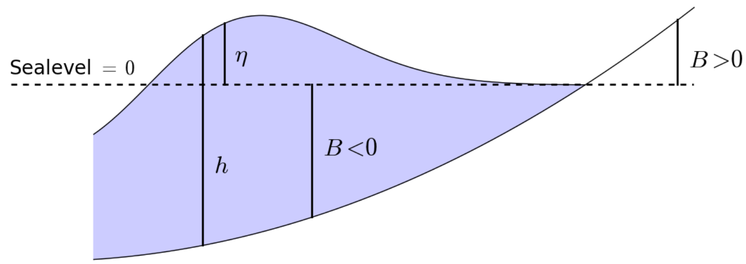

Figure 3: Illustration showing the notation used in the shallow water equations (1).

measured wave forms at many locations it is possible to construct an estimate of how the earth must have moved to produce this set of data. This relies ultimately on solving an inverse problem for the PDEs modeling wave motion in elastic materials [pointer to article on elasticity? Any article on seismology?]. Seismic inversions generally estimate the slip of the earth along the earthquake fault, which may be 10s of kilometers below the sea floor. Converting this slip on the fault plane to deformation of the sea floor requires solving another elasticity problem, whose solution is often approximated by the *Okada model*. This is based on the Green's function for the deformation of the boundary of an elastic half space caused by a delta function dislocation. Integrating this over a finite-sized patch of a fault plane gives an estimate of the resulting seafloor displacement.

While the results of seismic inversions are invaluable in modeling tsunamis, performing an accurate inversion requires collecting and processing a large amount of data and this may not be feasible in real time. In order to gather better information about tsunamis as they propagate, a number of pressure gauges have recently been deployed on the seafloor that are able to measure water pressure extremely accurately. From the hydrostatic pressure it is possible to estimate the depth of the water at these locations with enough precision to capture variations due to a tsunami passing by. Direct measurement of a tsunami at one or more of these gauges can then be combined with seismic models identifying the approximate source location and geophysical knowledge of the faults that are most likely to produce tsunamis. This information, together with accurate tsunami propagation models, can allow the solution of an inverse problem in order to estimate the seafloor deformation that caused the tsunami more accurately and quickly than is possible using seismic information alone.

3.3 Hazard modeling and mitigation

Real-time simulations of tsunamis are used to issue warnings, but tsunami modeling has many ongoing uses beyond this. Protecting communities requires adequate planning long before a tsunami takes place, and tsunami models are used to simulate the effect of tsunamis arising from hypothetical earthquake events. The results of such models can be used to determine what regions of a community are most at risk and what regions can be designated as safe zones for evacuation. Modeling the arrival time and pattern of the waves can be used in connection with traffic-flow models of evacuation. Some communities in tsunami-prone regions build sea walls or gates that can be closed for protection against tsunamis, or build "vertical evacuation structures" in regions where there is no easily accessible high ground for large-scale evacuation. These structures may take the form of multi-use buildings built to withstand tsunamis and tall enough that the upper floors are safe havens, or may consist of large berms that form artificial high ground. Designing such structures requires modeling the flow depth and often also the fluid velocities of hypothetical tsunamis.

Of course it is impossible to know exactly what the seafloor deformation will be for future earthquakes, but quite a bit is known about the major subduction zones and the likely location and magnitude of large earthquakes based on the geology and past history. There is always a question of how large a tsunami one should design for. Sometimes an estimate of the "credible worst case" tsunami for that location is used, but this may correspond to an event with very low probability of occurence that would require an enormous expenditure to protect against, money that might be better spent protecting against more likely events at additional locations. To better understand these tradeoffs, recently there has been increased interest in *probabilistic tsunami hazard assessment (PTHA)* in which a set of possible events are assigned probabilities, or an entire spectrum of possible events is assigned some probability density function, typically over a very high-dimensional stochastic space. The goal is then to obtain from this a probabilistic description of the resulting inundation patterns, flow depths, velocities, etc. This is a form of *uncertainty quantification (UQ)*, a rapidly growing field of importance in many fields of computational science where simulations are based on many uncertain inputs and the goal is a probabilistic description of the resulting outputs rather than a single simulation result. Applied mathematicians and statisticians have a large role to play in the development of new techniques to efficiently solve these problems. [Pointer to article on UQ?]

3.4 The study of past tsunamis and earthquakes

Another major use of tsunami modeling is the study of past tsunamis. A wealth of data has been collected following recent tsunami events by "tsunami survey teams" that measure inundation and runup along affected coasts. There is also data available from seafloor pressure gauges, tide gauges along the coast, and other data collection facilities. Models of the seafloor deformation produced by solving the source inversion problem can then be used as initial data for tsunami models and the computed results compared with measurements. Such studies are important in verifying that a tsunami model gives a sufficiently accurate approximation to a real tsunami that it can be used with confidence for warning or hazard mitigation purposes. [Pointer to article on Verification and Validation?] Validated models are also used in performing tsunami source inversion to estimate the seafloor deformation, and can give additional insight into the earthquake mechanism that is useful to seismologists. Tsunami models can also help explain unusual features of past events by providing a laboratory for exploring the fluid dynamics taking place during the event.

Tsunami models can also help reconstruct events that happened in the more distant past, for which there are no pressure gauge or tide gauge data and perhaps limited historical records of the regions inundated, or no human records in the case of prehistoric events or those that occurred on uninhabited coastlines. Luckily, for many events a geological record of the tsunami inundation is recorded in the form of *tsunami deposits*. As a tsunami approaches shore it typically becomes turbulent and picks up sediment from the seafloor, such as sand and marine microorganisms. This material is carried inland during the flooding stage and typically settles out of the flow as the flow decelerates and reverses, leaving behind a layer of deposits, often far inland. In tsunami-prone areas there are often many layers of tsunami deposits that have been built up over thousands of years, separated by layers of soil that slowly build up between tsunamis. Core samples or trenches can reveal many past events that can often be dated using radiocarbon dating of organic matter or interspersed tephra layers from known volcanic eruptions. The study of tsunami deposits is a major source of information about the magnitude, location, and recurrence times of past earthquakes. This information is critical in developing probabilistic models of tsunami or earthquake hazards, as well as to obtaining a better scientific understanding of earthquake processes. Numerical tsunami models can be used to help identify the location and magnitude of seafloor deformation that would lead to the patterns of inundation recorded by tsunami deposits. Models that include sediment erosion, transport, and deposition are also being used to better understand the fluid dynamics of the creation of tsunami deposits, which ultimately will lead to more accurate descriptions of the tsunamis that caused observed deposits.

4 Numerical modeling

Systems of nonlinear PDEs such as the nonlinear shallow water equations (1) typically cannot be solved exactly except for very simple cases, for example a one-dimensional wave on a linear beach. Realistic tsunami modeling always relies on numerical solution of the PDEs. This requires discretizing the equations in some manner: replacing the differential equations describing the continuum solution (defined for all (x, y, t) in some domain) by a finite set of discrete algebraic equations whose solution can be computed in finite time on a computer. There are many ways to do this, and general discussions of numerical solution of differential equations are given in [other articles??].

Finite difference methods are often used, in which a discrete grid is introduced consisting of a finite number of grid points (x_i, y_j) covering the domain, and the solution is approximated only at these points at a discrete set of times $t_0,\ t_1,\ t_2,\ \ldots$. Derivatives in the PDE are replaced by finite difference approximations based on the approximate solution at neighboring grid points, obtaining a discrete set of algebraic equations that can be solved on a computer. Another popular approach is to use a *finite volume method*, in which the domain is subdivided into a finite number of grid cells and the approximate solution consists of average values of the solution over each grid cell. Integrating the PDEs over a grid cell gives an expression for the time derivative of the cell average that can be used to update the cell averages from one time t_n to the next time t_{n+1}. To obtain better accuracy, methods are sometimes used in which the solution on each grid cell is approximated by a polynomial rather than by only the cell average (which can be interpreted as a constant function, or polynomial of degree 0, over each cell). In this case, the higher order coefficients of each polynomial must be updated from one timestep to the next. A method of this type that has recently become popular for tsunami modeling is the *Discontinuous Galerkin method*, in which the piecewise polynomial function obtained from the polynomials defined on each cell is not assumed to be continuous at the interface between one cell and its neighbor. The term "Galerkin" refers to a *finite element* approach to deriving equations for evolving the polynomial coefficients in time. [Pointers to other sections.]

4.1 Nonlinearity and shock formation

A prominent feature of nonlinear hyperbolic PDEs is that *shocks* can form in the solutions, which are discontinuities in the depth and velocity that can arise even from smooth initial conditions. [Article on shocks or nonlinear conservation laws?] As mentioned in Section 2, these correspond to hydraulic jumps or bores that are seen in tsunamis as they approach the shore. Sharp discontinuities are only an approximation to the true behavior, but often give a good approximation to the flow. Incorporating more accurate fluid dynamics models would lead to systems of PDEs that are much more computationally expensive to solve.

The presence of discontinuities in the solution can lead to difficulties in solving the PDEs numerically, since derivatives are infinite at a point of discontinuity, and finite difference approximations to derivatives generally diverge. This has lead to the increased popularity of both finite volume and Discontinuous Galerkin methods, which are better able to robustly capture discontinuities in the solution. Methods designed to do this well are often called *shock capturing methods*.

4.2 Inundation and the moving shoreline

Another computational challenge in modeling tsunamis, or any other geophysical flow over topography, is the need to handle the moving boundary of the flow at the shoreline. Many early tsunami models did not capture this moving boundary at all. Instead the equations were solved over a fixed domain defined by the original shoreline, and some boundary conditions imposed at this fixed boundary, such as an impermeable wall. While this cannot be used to model inundation directly, it could still give some indication of the tsunami runup based on recording the depth and velocities along this wall boundary. Other mathematical or physical models were then used to estimate inundation from these values.

Most tsunami models developed recently attempt to model inundation directly. For simple problems it may be possible to use a grid that moves with time so that one edge of the grid is always along the shoreline. For realistic problems this is generally infeasible since the shoreline can be very complex and can break into pieces as islands or isolated pools of water form. Most tsunami models instead use a fixed grid and implement some form of *wetting and drying algorithm* to keep track of which grid points or cells are dry ($h = 0$) and which are wet ($h > 0$). Standard approaches to approximating the PDEs typically break down near the shoreline and a major challenge in developing tsunami models is to deal with this case robustly and accurately, particularly since this is often the region of primary interest in terms of the model results.

4.3 Mesh refinement

Another challenge arises from the vast differences in spatial scale between the ocean over which a tsunmai propagates and a section of coastline such as a harbor where the solution is of interest. In the shoreline region it may be necessary to have a fine grid with perhaps 10 m or less between grid points in order to resolve the flow at a scale that is useful. It is clearly impractical and luckily also unnecessary to resolve the entire ocean to this resolution. The wave length of a tsunami is typically more than 100 km, so the grid point spacing in the ocean can be more like 1–10 km. Moreover we need even less resolution over most of the ocean, particularly before the tsunami arrives.

To deal with the variation in spatial scales, virtually all tsunami codes use unequally spaced grids, often by starting with a coarse grid over the ocean and then refining portions of the grid to higher resolution where needed. Some models only use static refinement, in which the grid does not change with time but has finer grids in regions of interest along the coast. Other computer codes use *adaptive mesh refinement*, in which the regions of refinement change with time, for example to follow the propagating tsunami with a finer grid near the wave peak than is used over the rest of the ocean, and to refine near the coastal region of interest only when the tsunami is approaching.

A related issue is the choice of time steps for advancing the solution. Stability conditions generally require that the time step multiplied by the maximum wave speed should be no greater than the width of a grid cell. This is because the explicit methods that are typically used for solving hyperbolic PDEs such as the shallow water equations update the solution in each grid cell based only on data from the neighboring cells in each time step. If a wave can propagate more than one grid cell in a timestep then the method becomes unstable. This necessary condition for stability is called the *CFL condition* after fundamental work on the convergence of numerical methods by Courant, Friedrichs, and Lewy in the 1920s. [Pointer to other entry?] For the shallow water equations the wave speed is $\sqrt{gh}$, which varies dramatically from the shoreline where $h \approx 0$ to the deepest parts of the ocean, where h can reach 10,000 m. Additional difficulties arise in implementing an adaptive mesh refinement algorithm: if the grid is refined in part of the domain by a factor of 10, say, in each spatial dimension, then typically the time step must also be decreased by the same factor. Hence for every time step on the coarse grid it is necessary to take 10 time steps on the finer grid, and information must be exchanged between the grids to maintain an accurate and stable solution near the grid interfaces.

4.4 Dispersive terms

In some situations, tsunamis are generated with short wavelengths that are not sufficiently long relative to the fluid depth for the shallow water equations to be valid. This most frequently happens with smaller localized sources such as a submarine landslide rather than large-scale earthquakes. In this case it is often still possible to use depth-averaged two-dimensional equations, but the equations obtained typically include additional terms involving higher order derivatives. These are generally dispersive terms that can better model the observed effect that waves with different wavelength propagate at different speeds.

The introduction of higher order derivatives typically requires the use of *implicit methods* to efficiently solve the equations, since the stability constraint for an explicit method would generally require a time step much smaller than desirable. Implicit methods result in coupled algebraic systems of equations, often nonlinear, for the solution at all of the grid points in each time step.

See [1] for a recent survey of tsunami sedimentology, and [2] for a general introduction to probabilistic modeling of tsunamis. Some detailed descriptions of numerical methods for tsunami simulation can be found, for example, in [3, 4, 5, 6, 7, 8]. The use of dispersive equations for modeling submarine landslides is discussed, for example, in [9].

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