

Finite Volume Methods for Irregular One-Dimensional Grids

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ABSTRACT. We consider an approach for hyperbolic conservation laws on irregular grids in which the flux is determined based on an average value of the approximate solution over an interval of fixed size h to the left and right of the interface, where h is some measure of the average or maximum grid spacing. This interval may overlap several grid cells and hence the stencil is enlarged so that the method remains stable with a time step chosen relative to h .

1. Introduction

We consider finite volume methods for the hyperbolic system of conservation laws

$$(1.1) \quad u_t + f(u)_x = 0,$$

in one space dimension on an irregular grid. For simplicity we discuss scalar equations although the methods generalize easily to systems. Let U_i^n be the finite volume approximation to the cell average

$$U_i^n \approx \frac{1}{h_i} \int_{x_{i-1/2}}^{x_{i+1/2}} u(x, t_n) dx,$$

where $h_i = x_{i+1/2} - x_{i-1/2}$. The finite volume method takes the form

$$(1.2) \quad U_i^{n+1} = U_i^n - \frac{k}{h_i} (F_{i+1/2}^n - F_{i-1/2}^n),$$

where k is the time step and $F_{i-1/2}^n$ is the numerical flux at the grid interface between cells $i-1$ and i (we will often omit the superscript n when it is clear

1991 *Mathematics Subject Classification.* Primary 65M06; Secondary 65M12, 76M20.

This work was supported in part by NSF Grants DMS-8657319 and ACS-8858101, AFOSR Contract 91-0063, and DOE Contract DEFG02ER25053.

The final version of this paper will be submitted for publication elsewhere.

from context). The simplest methods such as upwind or Lax-Wendroff are 3-point methods in which $F_{i-1/2} = \bar{F}(U_{i-1}, U_i)$ for some fixed function \bar{F} .

Our goal is to develop robust methods that compute smooth and accurate solutions with reasonable time steps even if the grid is highly nonuniform. This cannot be achieved with 3-point methods since the CFL condition requires that k satisfy

$$(1.3) \quad k \max_u |f'(u)| \leq \min_i h_i,$$

and so it is the *smallest* cell that limits the time step, which may then be unreasonable relative to the majority of cells.

2. The h -box Method

The basic idea of the class of methods we are now studying is to define the flux $F_{i-1/2}$ based on a standard numerical flux \bar{F} but applied to modified data $U_{i-1/2}^L$ and $U_{i-1/2}^R$ rather than to the cell values U_{i-1} and U_i . The modified data is obtained by the following steps:

- (i) Reconstruct some function $\tilde{u}(x)$ from the cell-average data $\{U_i\}$ (e.g., piecewise constant, piecewise linear, or higher order).
- (ii) Compute the average of $\tilde{u}(x)$ over virtual cells of fixed length h to the left and right of the interface $x_{i-1/2}$:

$$U_{i-1/2}^L = \frac{1}{h} \int_{x_{i-1/2}-h}^{x_{i-1/2}} \tilde{u}(x) dx \quad \text{and} \quad U_{i-1/2}^R = \frac{1}{h} \int_{x_{i-1/2}}^{x_{i-1/2}+h} \tilde{u}(x) dx.$$

Note that this is easily done in practice by using an interpolation of the primitive function $W_i = \sum_{j=0}^i U_j h_j \approx \int_0^{x_{i+1/2}} u(x) dx$.

Here, h is some measure of the average or maximum cell size on the grid. We refer to the interval $[x_{i-1/2}-h, x_{i-1/2}]$ as the h -box to the left of the interface $i-1/2$. After computing $U_{i-1/2}^L$ and $U_{i-1/2}^R$, we then set

$$F_{i-1/2} = \bar{F}(U_{i-1/2}^L, U_{i-1/2}^R)$$

and use the flux-differencing expression (1.2) to update U_i . On regular grids, with $h_i = h$, this is identical to the regular scheme.

Note that each h -box may overlap several grid cells, so that the method is no longer a 3-point method in general. Because the flux at $x_{i-1/2}$ is now computed based on information a distance h away on either side, the CFL condition now only requires that

$$(2.1) \quad k \max_u |f'(u)| \leq h,$$

and the methods are typically stable with this relaxed time step restriction. Note, however, that we still divide by h_i in (1.2), which could be much smaller than h . Stability is still maintained since for very small cells the h -boxes used

to compute $F_{i-1/2}$ and $F_{i+1/2}$ overlap nearly completely, so that these fluxes are nearly equal and in fact $F_{i+1/2} = F_{i-1/2} + O(h_i)$ as $h_i \rightarrow 0$. (See [1].)

Besides increasing the time step, this approach yields much smoother solutions than other methods on highly nonuniform grids. The use of fixed-size h -boxes to compute the fluxes introduces a projection onto a uniform grid.

3. Numerical Examples

The upwind method in the case $f'(u) > 0$ has the flux

$$\bar{F}(U^L, U^R) = f(U^L).$$

The standard upwind method on a nonuniform grid takes the form

$$U_i^{n+1} = U_i - \frac{k}{h_i} (f(U_i) - f(U_{i-1}))$$

and is stable only if (1.3) is satisfied. Moreover, a truncation error analysis shows that this method is formally not even consistent unless the grid is nearly uniform in the sense that $h_i/h_{i-1} = 1 + O(h)$ as $h \rightarrow 0$. In fact it can be shown [4] (see also [3], [5]) that the method is convergent and even first-order accurate in spite of this, on arbitrary grids, but the error is typically not smooth. Figure 1 shows an example calculation for Burgers' equation, $f(u) = u^2/2$, with data $u(x, 0) = \frac{1}{2}(1 - \sin 2\pi x)$ at time $t = 0.2$, which gives a smooth solution. The grid has random cell sizes h_i uniformly distributed between $h/10$ and h , with $h = 1/25$. There are 48 grid cells. In Figure 1a, the standard upwind method is used with time step $k = h/10$, as required for stability.

Figures 1b and 1c show the result obtained on the same problem when h -boxes are used to define the values $U_{i-1/2}^L$ and then

$$U_i^{n+1} = U_i - \frac{k}{h_i} (f(U_{i+1/2}^L) - f(U_{i-1/2}^L)).$$

In this case, $k = h$ is used on the same random grid as before. Figure 1b shows results when $\tilde{u}(x)$ is taken to be the piecewise constant function with value U_i in the i th cell. Figure 1c shows the results obtained if $\tilde{u}(x)$ is the piecewise linear function

$$\tilde{u}(x) = U_i + (x - x_i)(U_i - U_{i-1})/h_{i-1/2} \quad \text{on cell } [x_{i-1/2}, x_{i+1/2}].$$

Here, $h_{i-1/2} = (h_{i-1} + h_i)/2$ and $x_i = (x_{i-1/2} + x_{i+1/2})/2$, the cell midpoint.

Figure 1d shows results when the Lax-Wendroff flux is used together with piecewise linear reconstruction for $\tilde{u}(x)$. The Lax-Wendroff flux for $f'(u) \geq 0$ is given by

$$\bar{F}(U^L, U^R) = \frac{1}{2}(f(U^L) + f(U^R)) - \frac{k}{2h} \left(\frac{(f(U^R) - f(U^L))^2}{U^R - U^L} \right).$$

Grid refinement studies show that this method is second-order accurate (for smooth solutions) in h , while all of the previous methods are first-order accurate.

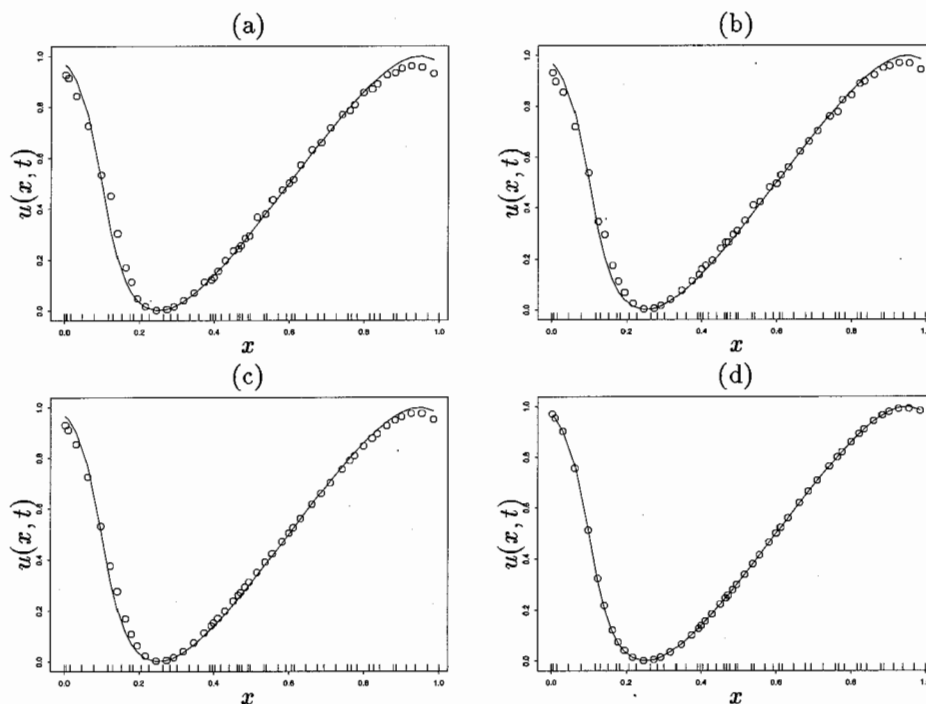


FIGURE 1. Results on a random grid for Burgers' equation with a smooth solution. (a) The standard upwind method with $k = h/10$. (b) The h -box upwind method with piecewise constant $\tilde{u}(x)$. (c) The h -box upwind method with piecewise linear $\tilde{u}(x)$. (d) The h -box Lax-Wendroff method with piecewise linear $\tilde{u}(x)$.

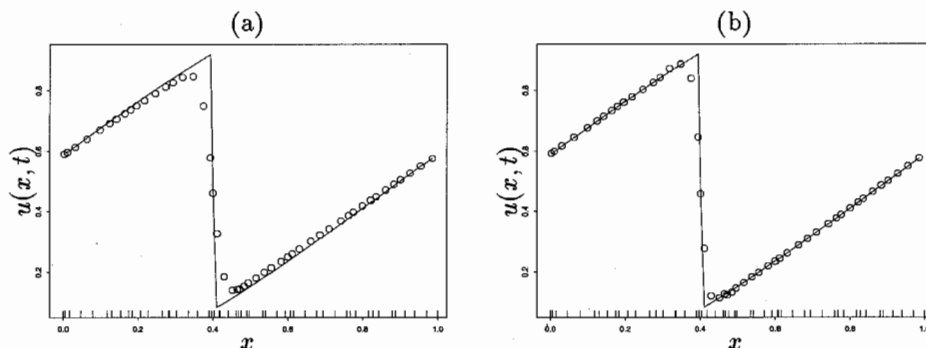


FIGURE 2. Results on a random grid for Burgers' equation with a discontinuous solution. (a) The h -box upwind method with piecewise linear $\tilde{u}(x)$. (b) The h -box high resolution flux limiter method with piecewise linear $\tilde{u}(x)$.

4. Extensions

The ideas presented here can be extended to situations where the basic flux \bar{F} depends on more than two points, and in particular to flux-limiter methods for the high-resolution computation of shock waves. This will be presented in more detail elsewhere. As an example, Figure 2 shows the same techniques as Figure 1c and 1d at time $t = 0.8$, after a shock has developed.

There remain some problems, particularly when small cells are located near a transonic rarefaction. Several interfaces may appear to be transonic points because their h -boxes overlap the transonic region, causing large inaccuracies and oscillations. This can be corrected by reducing the size of the h -boxes near transonic points, where the propagation speed is smaller.

Although we consider only one-dimensional grids here, the original motivation for this problem comes from problems in more than one dimension, where nonuniform grids arise more commonly. The idea of creating h -boxes that overlap several grid cells as a way to compute stable fluxes was introduced by Berger and LeVeque[1], [2] as a way to deal with small cells created near the boundary when a Cartesian grid is used on an irregular region. The same idea could be used at interfaces between different grids with a composite grid method or perhaps at all cell interfaces on an unstructured grid. A more complete study of such methods is underway. Even in one space dimension these ideas could find application in conjunction with moving-mesh or front-tracking algorithms where nonuniformities in the grid are created.

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