Proc. Third Int'l Conf. Hyperbolic Problems,
Uppsala, Sweden, June, 1990.

Cartesian Meshes and Adaptive Mesh Refinement for Hyperbolic Partial Differential Equations

Marsha Berger	*Randall LeVeque*
Courant Institute	Math Department
251 Mercer St.	University of Washington
NY NY 10012	Seattle, WA 98125

ABSTRACT

We describe a Cartesian mesh algorithm with adaptive mesh refinement for computing fluid flows in complicated geometries. Stable boundary conditions are needed at the irregular cells where the Cartesian mesh intersects the body. We develop a difference scheme that is stable even when these irregular cells are orders of magnitude smaller than the regular cells. We illustrate its performance with some computational examples solving the two dimensional Euler equations for inviscid flow.

Introduction

We describe a Cartesian mesh method to solve fluid flow problems in complicated geometries. In this approach, we keep a uniform rectangular (Cartesian) grid and allow the solid boundary to intersect the grid cells in an essentially arbitrary way. Cartesian meshes are an appealing way to simplify the grid generation problem for complex domains. Multiply connected domains and irregular geometries are only slightly more complicated than a simple domain.

Cartesian meshes have by and large been overlooked in favor of body-fitted meshes or the more recently popular unstructured meshes, but they deserve much more attention. Cartesian meshes have the advantage of allowing the use of high resolution methods for shock capturing that are difficult to develop on unstructured grids. They also allow for efficient implementation on vector computers without using gather-scatter operations (except at boundary cells). They incur little computational or memory overhead since there are no metric terms and they use far fewer pointer arrays than their unstructured counterparts. Among the few references on Cartesian mesh methods are [Clarke, Salas and Hassan; Choi and Grossman].

The major technical issue in Cartesian mesh methods is the small cell problem. Arbitrarily small cells arise at the edge of the domain where the grid intersects a body. Stable, accurate and conservative difference schemes are needed for these cells. Moreover, the time step for a time-accurate computation should still be based on the volume of the regular cells away from the body, and not restricted by small boundary cells. Previous efforts to use Cartesian meshes have merged these small cells together until a cell with sufficient volume for stability is obtained. Clearly this loses resolution. In addition, if cell size is not taken into account in implementing finite difference schemes on irregular grids, a second order scheme can lose one or two orders of accuracy.

Our treatment of boundaries can be combined naturally with our adaptive refinement strategy using locally uniform meshes. We retain the advantages (efficiency and accuracy) of uniform grids and are able to resolve fine scale flow features induced by complex geometries. We are using the adaptive mesh refinement algorithm (AMR) described in [Berger and Colella] to achieve accuracy comparable to the body-fitted meshes, where grid points can be bunched in an a priori manner to improve the accuracy of the solution.

A more complete description of our approach to developing stable boundary conditions for irregular cells is described in [Berger and LeVeque]. Here, we only sketch the main ideas, and present computational examples for two dimensional time dependent flow in several different geometries.

One Dimensional Model Problem

We motivate the basic approach in one dimension, where we solve the equation $u_t + f(u)_x = 0$ on a uniform grid except for one small cell in the middle (see Figure 1). Let h be the cell size of the uniform grid, and αh the small cell size, $0 \le \alpha \le 1$. We use an explicit finite volume scheme to update all cells,

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{h}(f_{i+1/2} - f_{i-1/2})$$

at the regular cells, $i \neq 0$, and

$$u_0^{n+1} = u_0^n - \frac{\Delta t}{\alpha h}(f_{1/2} - f_{-1/2})$$

at the small cell. We want to define the fluxes $f_{\pm 1/2}$ so that the overall scheme is stable as $\alpha \to 0$.

Typical flux functions for the regular cells include Godunov's method, which for a scalar equations with $f'(u) \le 0$ is just upwind differencing, with

$$f_{i+1/2} = F(u_i, u_{i+1}) = f(u_{i+1})$$

and the second order Lax Wendroff scheme

$$F(u_i, u_{i+1}) = \frac{(u_i + u_{i+1})}{2} + \frac{\Delta t}{2h}(f(u_{i+1}) - f(u_i)).$$

However, these definitions must be modified at the small cell, since if we define $f_{1/2} = F(u_0, u_1)$, the resulting scheme loses accuracy and becomes unstable for small α. Our approach is to define a new state

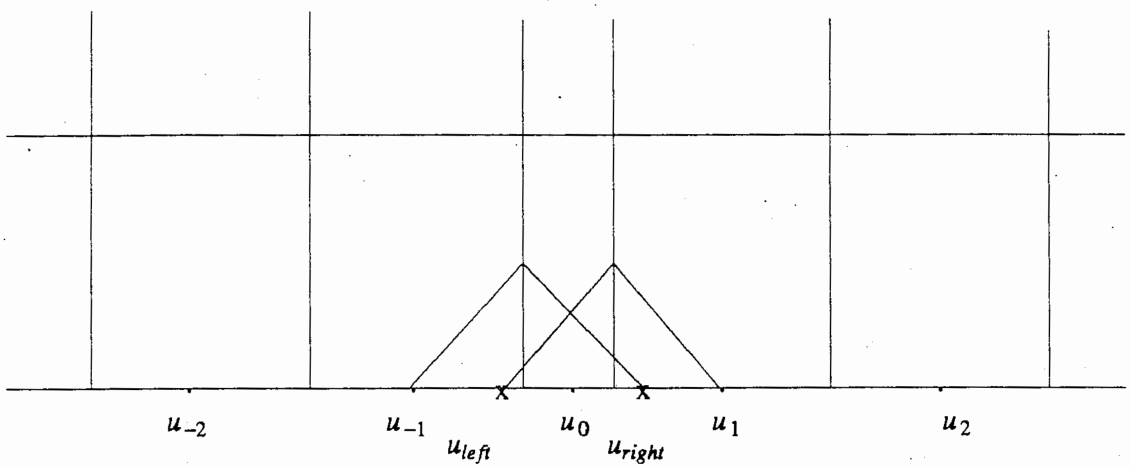


Figure 1 Special flux formulas are needed at the edges of the small cell to maintain stability and accuracy.

u_{left} and let

$$f_{1/2} = F(u_{left}, u_1).$$

Here, u_{left} is an approximation to u at distance $h/2$ from the interface. For example, u_{left} might be obtained by linear interpolation between u_{-1} and u_0,

$$u_{left} = \frac{2\alpha u_0 + (1-\alpha) u_{-1}}{1+\alpha}.$$

Note that $u_{left} \to u_0$ as $\alpha \to 1$, and $u_{left} \to u_1$ as $\alpha \to 0$. In either of these limits the grid becomes regular again and the difference scheme reverts to the uniform scheme.

Similarly, we define a flux $f_{-1/2}$ using the left state u_{-1} and a new right state u_{right} defined by interpolating u to a point a distance $h/2$ to the right of this interface, e.g.

$$u_{right} = \frac{2\alpha u_0 + (1-\alpha) u_1}{1+\alpha}.$$

We then set

$$f_{-1/2} = F(u_{-1}, u_{right}).$$

It can be shown for the equation $u_t = u_x$ that if u_{left} and u_{right} are obtained by linear interpolation then the resulting scheme is stable as $\alpha \to 0$, using a time step Δt that satisfies the CFL condition for the regular grid, $\frac{\Delta t}{h} \leq 1$. This results holds for both upwind differencing and Lax Wendroff. However, for Lax Wendroff, a more accurate procedure would be to use quadratic interpolation for u_{left} and u_{right}. In this case, a combination of theoretical and numerical results show that if the additional point used in the interpolation is the upwind point then the scheme is stable as $\alpha \to 0$, but if the downwind point u_{-1} is used , then the CFL limit is reduced to 1/2.

Stable Difference Equations for Two Dimensional Irregular Cells

For two dimensional calculations, we again need a new difference scheme to compute fluxes at the edges of the irregular cells adjacent to the boundary. The boundary of a solid body is represented by a piecewise linear segment in each cell, so that the irregular cells can have either 3, 4 or 5 sides. Our approach uses locally normal and tangential coordinate directions to define left and right states for a Riemann problem at each cell edge. This fits naturally with the MUSCL scheme used in the interior of the domain in our calculations. However, we generalize the pointwise approach in the one dimensional case, and use aconservative averages of the solution over a box a distance h in the appropriate direction away from a cell edge.. For example, in Figure 2 the state q_l is obtained using an area-weighted average of the values $u_{i,j-1}$ and $u_{i,j}$ that intersect the box from the regular grid. In an analogous way we obtain the state q_r. These values are then rotated into a frame of reference that is tangent to the boundary, and a one dimensional Riemann problem in the tangential direction is solved. This gives the flux f_ξ. This procedure is repeated for the dashed boxes in Figure 2 that are normal to the boundary, giving a flux f_η in the η direction. (The part of the box that lies outside the domain is interpolated from $\bar{q}_k$, see Figure 3). The final value of the flux at the vertical interface is a linear combination $f_\xi \cos\theta + f_\eta \sin\theta$, where θ is the angle the boundary makes with the grid. For a boundary with curvature, we determine these directions using the boundary segment of the cell with the smaller area adjacent to the interface. This helps retain stability for the

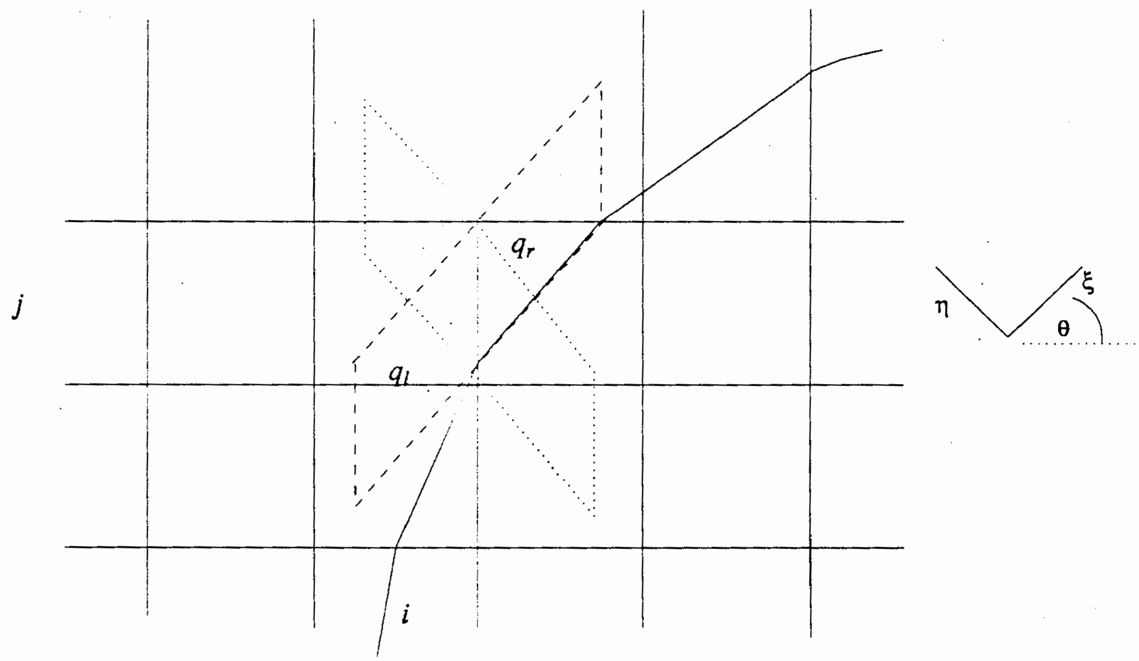


Figure 2 shows a schematic of the rotated difference scheme used to define the vertical flux.

smaller cells, by maintaining a certain cancellation property of our flux definitions, described more completely in [Berger and LeVeque]. Related work using rotated difference schemes has been done by [Jameson; Davis; Levy, Powell and van Leer].

At the solid wall boundary itself, the flux can be determined more simply, using only boxes normal to the boundary as shown in Figure 3. First we obtain a value q_k for the box interior to the domain, using area weighted averages, and rotate the velocities into the boundary coordinate frame. A boundary Riemann problem is solved between q_k and $\bar{q}_k$ (with negated normal velocity), to satisfy the the boundary conditions of no normal flow.

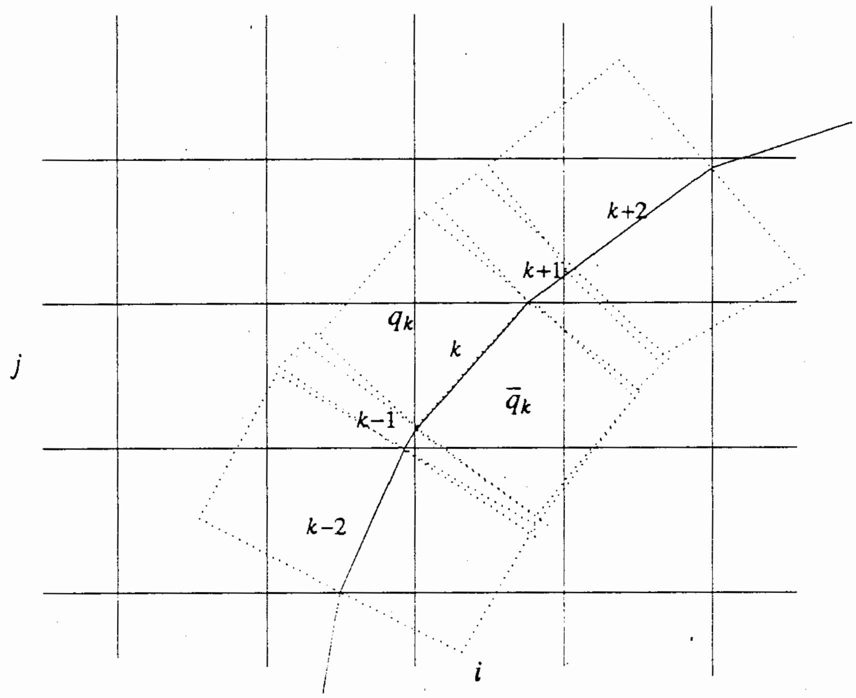


Figure 3 indicates the scheme used to determine the boundary flux.

In this case, if the solid wall boundary happens to align with the Cartesian grid, the scheme reverts to the usual first order Godunov method. To improve the scheme to second order, following the MUSCL approach as described in [Colella], we need to introduce limited slopes in the solution reconstruction phase, and tangential derivatives for predicting states at the cell edges. These steps are also necessary to improve the stability limit for Godunov's method from 1/2 to 1. Work on these improvements is continuing. Referring to Figure 3, we add a tangential derivative f_ξ to the state q_k for the normal Riemann problem, with $f_\xi = f(u(k,k+1)) - f(u(k-1,k)) / l_k$, where l_k is the length of the k^{th} boundary segment. The state $u(k,k+1)$ comes from solving a Riemann problem in the tangential direction at the interface between cells k and $k+1$. As before, the stencil for this Riemann problem must be enlarged beyond the adjacent cells to maintain stability. For example, the right state at this interface is not just the value q_{k+1}, but a linear combination of the solution in several boxes, q_{k+1} and q_{k+2}, up to a distance h away from the interface. The left state at this right interface can be taken to be the value q_k since the length of that cell's boundary segment is larger than h. This same procedure is used to include tangential derivatives in the normal box Riemann problems for interior cell edges. This procedure alone improves the CFL limit to 1. It remains to incorporate monotonized slopes into the scheme in order to achieve second order accuracy.

While the overall scheme at the boundary involves twice as many Riemann problems as the ordinary MUSCL scheme, it is fully vectorizable. The coefficients in the interpolations for the left and right states are fixed for the duration of the integration, and are not dependent of the properties of the solution at each step. In numerical experiments in two dimensions, this scheme remains stable for cell areas that are orders of magnitude smaller than the regular cell areas (down to the round-off level). In essence, our method can be viewed as a technique for defining fluxes on an irregular grid by a very local mapping to a regular grid. This viewpoint may prove useful in defining higher order methods on unstructured grids.

Computational Example

We illustrate this by computing time dependent flow around a cylinder. The initial conditions are an incident shock traveling at Mach 2.81. We use a simple MUSCL scheme to advance the flow field in the interior of the domain. Figure 4 shows a contour plot of the flow field, as well as a plot of density as a function of arclength around the cylinder. The only cells drawn on the contour plot are the irregular cells from the Cartesian grid that intersect the body. Note the smoothness of the arclength plot despite the irregularity of the grid around the body. This example was computed using the local mesh refinement of [Berger and Colella]; the location of the rectangular fine grids is indicated on the contour plot as well.

References

M. Berger and P. Colella, "Local Adaptive Mesh Refinement for Shock Hydrodynamics". J. Comp. Phys. 82, 1989.

M. Berger and R. LeVeque, "Stable Boundary Conditions for Cartesian Grid Calculations". Proc. Symposium on Computational Technology for Flight Vehicles, Pergamon Press, Nov. 1990, to appear. ICASE Report No. 90-37, May, 1990.

S. Choi and B. Grossman, "A Flux-Vector Split, Finite Volume Method for Euler's Equations on Non-Mapped Grids", AIAA Paper 88-0227, Reno, Nevada.

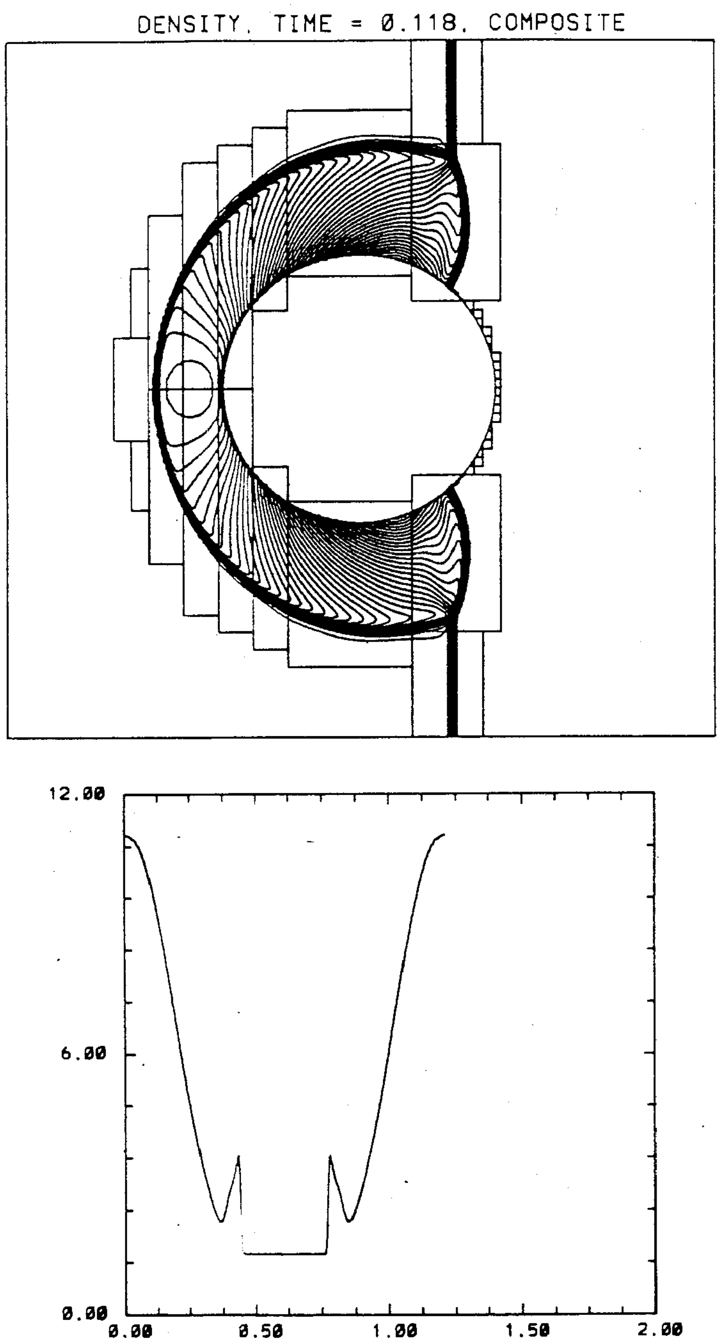
D. Clarke, M. Salas and H. Hassan, "Euler Calculations for Multielement Airfoils Using Cartesian Grids", AIAA Journal 24(3), 1986.

P. Colella, "Multidimensional Upwind Methods for Hyperbolic Conservation Laws", J. Comp. Phys. 87, 1990.

S. Davis, "A Rotationally Biased Upwind Difference Scheme for the Euler Equations", J. Comp. Phys. 56, 1984.

D. Levy, K. Powell, and B. van Leer, "An Implementation of a Grid Independent Upwind Scheme for the Euler Equations", AIAA Paper 89-1931-CP, Buffalo, NY.

A. Jameson, "Iterative Solution of Transonic Flows over Airfoils and Wings, Including Flows at Mach 1", Comm. Pure Appl. Math 27, 1974.



(a)

(b)

Figure 4 (a) Density contours of the flow around the cylinder. (b) Density around the boundary of the cylinder.