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Intermediate Boundary Conditions for LOD, ADI and Approximate Factorization Methods

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Abstract. A general approach to determining the correct intermediate boundary conditions for dimensional splitting methods is presented and illustrated. The intermediate solution U^* is viewed as a second order accurate approximation to a modified equation. Deriving the modified equation and using the relationship between this equation and the original equation allows us to determine the correct boundary conditions for U^*. To illustrate this technique, we apply it to LOD and ADI methods for the heat equation in two and three space dimensions. The approximate factorization method is considered in slightly more generality.

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1. Introduction

When solving time-dependent partial differential equations numerically, it is often desirable to use implicit methods for reasons of numerical stability. This is particularly true for parabolic equations where the time step restriction is severe for explicit methods. In more than one space dimesion, however, implicit methods lead to linear systems of equations with complicated structure which cannot be solved efficiently by direct methods.

Consequently, a wide variety of dimensional splitting or fractional step methods have been used. In these methods, a single multidimensional implicit time step is replaced by a sequence of steps, each of which is implicit in only one coordinate direction. In that direction, the equations can be solved along one line of gridpoints at a time, giving a banded system of equations which can be easily solved. The reader is referred to [2], [4], [10], [13] for an introduction to many of the methods used in practice.

One difficulty which has sometimes plagued these methods and frequently caused confusion among users is the proper specification of boundary conditions for the intermediate solutions which arise between the individual steps of this sequence. The purpose of this paper is to show how the proper boundary conditions can frequently by determined quite simply and logically by considering the "modified equation" (or "model equation") corresponding to each finite difference approximation in the sequence.

We will illustrate this technique by considering several fractional step methods for the heat equation in two space dimensions: the locally one dimensional (LOD) method, the alternating-direction implicit (ADI) methods of Peaceman-Rachford[11] and Douglas-Gunn[3], and the approximate factorization (AF) method of Beam and Warming[1]. General boundary conditions involving normal derivatives are handled.

We have chosen a simple problem and well-known fractional step methods in order to illustrate the technique as clearly as possible. The same ideas used here can be applied to other methods and more complicated equations. The extension to methods involving more than two fractional steps is illustrated by considering the LOD method in three space dimensions in Section 6.

Although here we only discuss dimensional splitting methods, the same approach can be used to determine the correct intermediate boundary conditions for any additive splitting method. Other contexts

where these ideas have already been applied include the splitting of hyperbolic systems into subproblems with disparate wave speeds[8], [9], the splitting of convection-diffusion equations into hyperbolic and parabolic subproblems[8], and splittings of the incompressible Navier-Stokes equations[7].

For most of this paper we will consider the two-dimensional heat equation,

$$u_t = u_{xx} + u_{yy} = Lu \qquad \text{in } \Omega = [0,1]\times[0,1] \tag{1.1}$$

with boundary conditions on $\partial\Omega$. Along the boundary $x = x_b$ ($x_b = 0$ or 1), for example, we will consider either the Dirichlet condition

$$u(x_b,y,t) = g(y,t) \tag{1.2a}$$

or the more general Robbins condition

$$\alpha(y,t)u(x_b,y,t) + \beta(y,t)u_x(x_b,y,t) = \gamma(y,t). \tag{1.2b}$$

A fractional step method for this problem typically has the form

$$Q_x U^*_{ij} = P_1 U^n_{ij} \tag{1.3a}$$

$$Q_y U^{n+1}_{ij} = P_2 U^*_{ij}. \tag{1.3b}$$

Here the P's and Q's are spatial difference operators, with Q_x and Q_y involving differences in only the x- and y-directions respectively. U^n_{ij} represents the numerical approximation to the solution $u(x_i,y_j,t_n)$. For simpicity, we take equal mesh widths h in the x and y directions, so $x_i = ih$ and $y_j = jh$ for $i,j = 0,1,\ldots,N$, where $h = 1/N$. The time step is denoted by k, so $t_n = nk$, $n = 0,1,\ldots$.

The problem is to determine appropriate boundary conditions for the "intermediate solution" U^*_{ij} which arises in (1.3). We typically require boundary data U^*_{bj}, $b = 0, N$ for (1.3a). We may also require U^*_{ib}, $b = 0, N$, for (1.3b) if P_2 involves y-differences.

Typically, (1.3) gives a second order accurate approximation U^{n+1}, while the individual steps (1.3a) and (1.3b) are not second order accurate methods for the original equation. Hence the intermediate solution U^* is nonphysical and does not correspond to the true solution at any intermediate time. This makes it difficult to determine a priori the proper boundary conditions.

The approach we will take consists of determining the modified equation for (1.3a), the differential equation which *would* be solved to second order accuracy if we iterated with the scheme

$$Q_x U_{ij}^{n+1} = P_1 U_{ij}^n.$$

The modified equation is derived by replacing the difference operators by expansions of differential operators. This expansion can be truncated at an appropriate point to give a differential equation.

We denote the modified equation by

$$u_t^* = L^* u^* \tag{1.4a}$$

where L^* is a differential operator involving only spatial derivatives. Similarly, we can derive the modified equation for (1.3b):

$$u_t^{**} = L^{**} u^{**}. \tag{1.4b}$$

By truncating the expansions in each modified equation appropriately, we will have

$$L = L^* + L^{**}, \tag{1.5}$$

an additive splitting of the original operator.

There is an obvious splitting of L into one dimensional operators, namely

$$L^* = \partial_x^2, \qquad L^{**} = \partial_y^2. \tag{1.6}$$

This is precisely the splitting used by the LOD method discussed in Section 2. The other methods, however, use more complicated splittings of the operator L.

Once the modified equation (1.4a) has been determined, we can view (1.3a) as taking a single time step on the equation (1.4a) with initial conditions

$$u^*(x,y,t_n) = u(x,y,t_n) \quad \forall\ x,y\ \in\ \Omega. \tag{1.7}$$

Here we assume $U_{ij}^n = u(x_i,y_j,t_n)$. If the solution to (1.4a), (1.7) is denoted by $u^*(x,y,t)$ for $t \geq t_n$, then

$$U_{ij}^* = u^*(x_i,y_j,t_n+k) + O(k^3) \tag{1.8a}$$

since (1.3a) is second order accurate on (1.4a). One approach to specifying boundary conditions for U^* is to simply employ a one-sided, second order accurate, finite difference approximation to (1.4a). Then we are solving the same equation at the boundary as in the interior, but with a one-sided scheme instead of

the (presumably centered) scheme (1.3a).

The problem with this approach is that it does not make any use of the specified boundary conditions (1.2). We can often do better by determining the boundary behavior of $u^*(x,y,t)$ in terms of the (given) boundary behavior of u. If we succeed in determining $u^*(x,y,t_n+k)$ to $O(k^3)$ for points along the boundary, this gives us the proper specification of U^*.

Actually, it usually suffices to have boundary data which are one order of accuracy lower than the interior scheme, i.e., we only need to insure that

$$U^*_{ij} = u^*(x_i,y_j,t_n+k) + O(k^2) \tag{1.8b}$$

at boundary points to maintain second order accuracy. For hyperbolic equations this is a result of Gustafsson[6]. For the heat equation it is a simple consequence of linearity and the maximum principle. However, in practice it is found that such boundary conditions can lead to a large increase in the error constant. Although the results are second order accurate as the mesh is refined, the errors on any particular mesh may be an order of magnitude larger than necessary. This can make a significant difference in the effort required to solve a multidimensional problem to a specified accuracy, and so we recommend using boundary conditions with $O(k^3)$ accuracy whenever possible. Some examples of the resulting increase in accuracy will be seen in later sections.

To determine the behavior of u^* along the boundary, consider a typical point along one of the boundaries $x = x_b$. Expanding $u^*(x_b,y,t_n+k)$ about time t_n gives

$$\begin{aligned} u^*(x_b,y,t_n+k) &= u^*(x_b,y,t_n) + ku^*_t(x_b,y,t_n) + \frac{1}{2}k^2u^*_{tt}(x_b,y,t_n) + \cdots \\ &= u^* + kL^*u^* + \frac{1}{2}k^2M^*u^* + \cdots . \end{aligned} \tag{1.9}$$

We obtain M^* by differentiating (1.4a) with respect to t and replacing time derivatives of u^* on the right hand side by L^*u, so that M^* involves only *spatial* derivatives. For the linear equation considered here we have $M^* = (L^*)^2$, but the same process can be applied even in nonlinear problems.

It is important that the final expression in (1.9) involves only spatial derivatives of u^* at time t_n. (We make the convention throughout this paper that if a function occurs with no arguments on the right hand side of such an expression, the point (x_b,y,t_n) is assumed.) According to the initial conditions (1.7), $u^* = u$ at time t_n and so $L^*u^* = L^*u$ and similarly for any spatial operator (N.B. this is valid only at time

and expand $u^*(x_b,y,t_n+k)$ about (x_b,y,t_n) to obtain

$$\begin{aligned} u^*(x_b,y,t_n+k) &= u^*(x_b,y,t_n) + ku_t^*(x_b,y,t_n) + \frac{1}{2}k^2u_{tt}^*(x_b,y,t_n) + \cdots \\ &= u^* + ku_{xx}^* + \frac{1}{2}k^2u_{xxxx}^* + \cdots \\ &= u + ku_{xx} + \frac{1}{2}k^2u_{xxxx} + \cdots \end{aligned} \tag{2.3}$$

which corresponds to (1.10). We now use the original equation (1.1) to solve for

$$u_{xx} = u_t - u_{yy}$$

and

$$u_{xxxx} = u_{tt} - 2u_{tyy} + u_{yyyy}.$$

This allows us to rewrite the boundary data (2.3) in terms of t- and y-derivatives of u along the boundary $x = x_b$:

$$\begin{aligned} u^*(x_b,y,t_n+k) &= u + k(u_t - u_{yy}) + \frac{1}{2}k^2(u_{tt} - 2u_{tyy} + u_{yyyy}) + \cdots \\ &= u(x_b,y,t_n+k) - ku_{yy} + \frac{1}{2}k^2(u_{yyyy} - 2u_{tyy}) + \cdots \\ &= [I - k\partial_y^2 + \frac{1}{2}k^2\partial_y^4 + \cdots]\, u(x_b,y,t_n+k). \end{aligned} \tag{2.4}$$

This can also be obtained by integrating (1.4b) backwards in time from U^{n+1}; see Section 6 for [illegible].

If the original problem specified Dirichlet boundary conditions, $u(x_b,y,t) = g(y,t)$, (2.4) immediately provides the appropriate boundary conditions for U^*. Based on (1.8a), we can take

$$U_{bj}^* = g(y_j,t_n+k) - kg_{yy}(y_j,t_n+k) + \frac{1}{2}k^2g_{yyyy}(y_j,t_n+k). \tag{2.5}$$

With the Robbins condition (2.2b), appropriate boundary conditions are still easy to derive if α and β are constant, so (2.2b) is

$$\alpha u(x_b,y,t) + \beta u_x(x_b,y,t) = \gamma(y,t). \tag{2.6}$$

Differentiating (2.4) with respect to x and taking the linear combination $\alpha u^* + \beta u_x^*$ gives the boundary condition

$$\alpha u_x^*(x_b,y,t_n+k) + \beta u_x^*(x_b,y,t_n+k) = \gamma(y,t_n+k) - k\gamma_{yy}(y,t_n+k) + \frac{1}{2}k^2\gamma_{yyyy}(y,t_n+k). \qquad (2.7)$$

If α and β are not constant then the situation is more complicated since, for example, $\alpha u_{yy} + \beta u_{xyy} \neq \gamma_{yy}$. Based on the second line of (2.4), we can still obtain a condition of the form

$$\alpha u^* + \beta u_x^* = \gamma(y,t_n+k) - k(\alpha u_{yy} + \beta u_{xyy}) + \frac{1}{2}k^2(\alpha u_{yyyy}+\beta u_{xyyyy} - 2\alpha u_{yy} - 2\beta u_{xyy}), \qquad (2.8)$$

but now the derivatives of $u(x_b,y,t_n)$ on the right hand side must be replaced by finite difference approximations based on U_{ij}^n. Unless this is done carefully, numerical instabilities can arise. This is discussed in more detail in Section 4.

To demonstrate the validity of the boundary conditions derived here, we present the results of two numerical experiments with Dirichlet conditions. We first consider the problem (1.1) with exact solution

$$u(x,y,t) = \sin(x/2 + y)e^{-1.25t} \qquad (2.9)$$

and specify boundary conditions obtained by evaluating (2.9) along the boundaries, e.g., along $x = 0$,

$$g(y,t) = \sin(y)e^{-1.25t}. \qquad (2.10)$$

We compare the results obtained with three different choices of boundary conditions U_{bj}^*. The first of these is simply

$$U_{bj}^* = g(y_j,t_n+k/2). \qquad (2.11)$$

This has been recommended by some authors, but comparing this to (2.5) shows that it differs by $O(k)$. We thus expect this boundary condition to destroy the second order accuracy. This is confirmed by the results of Table 2.1. Here we have displayed the errors at time $t = 0.75$ measured in both the L_∞- and L_2-norms. We have also computed the L_2-rate of convergence, obtained by comparing the errors with different values of k (we always take k/h fixed as the mesh is refined and $h = 1/N$). The L_2-rate is defined by $\log(e_2/e_1) / \log(k_2/k_1)$, where e_1 and e_2 are the L_2-errors obtained with time steps k_1 and k_2 respectively.

The second boundary condition used is

$$U_{bj}^* = g(y_j,t_n+k) - kg_{yy}(y_j,t_n+k), \qquad (2.12)$$

which was derived by Dwoyer and Thames[4] using the "method of undetermined functions." In this approach, one inserts undetermined constants in place of the unknown boundary values and combines the

conditions than directly discretizing (4.15j. For completeness we present the details of our AF implementation in Section 5.

As a numerical experiment, we compare (4.17) to the more naive conditions

$$\alpha^{n+\frac{1}{2}}\delta^* + \beta^{n+\frac{1}{2}}\delta_\tau^* = (\gamma^{n+\frac{1}{2}} - \gamma^n) - (\alpha^{n+\frac{1}{2}} - \alpha^n)u \tag{4.18}$$

which do not contain the $O(k^2)$ correction. We again use the heat equation (1.1) with solution

$$u(x,y,t) = \sin(x+y-1)\, e^{-2t}. \tag{4.19}$$

For simplicity, we specified Dirichlet boundary conditions along three boundaries and the Robbins condition

$$\alpha(y,t)u(1,y,t) + u_x(1,y,t) = \gamma(y,t)$$

only along the boundary $x = 1$. We take

$$\alpha(y,t) = \cos(y)\, e^t, \qquad \gamma(y,t) = \frac{1}{2}\sin(2y)\, e^{-t} + \cos(y)\, e^{-2t}.$$

Table 4.2 gives a comparison of results obtained at $t = 0.3$ with three different combinations of boundary conditions:

a) (4.13) at $x = 0$ and (4.18) at $x = 1$,

b) (4.12) at $x = 0$ and (4.18) at $x = 1$, (4.20)

c) (4.12) at $x = 0$ and (4.17) at $x = 1$.

As expected, all three combinations give second order accuracy, but including the correct $O(k^2)$ terms improves the error constant. The combination (4.20c) gives errors about 10 times smaller than (4.20a). In this experiment we have used the parameters $\theta = 1$, $\xi = 1/2$, $\omega = 2/3$ for improved stability.

5. Implementation of the AF Method

We include a detailed discussion of our implementation of (4.5) since in performing the experiments presented above it was found that both the accuracy and stability of the scheme were greatly affected by the manner in which the boundary conditions were imposed.

For each fixed $j = 1, 2, \ldots, N-1$, (4.5a) gives rise to a tridiagonal system of equations of the form

$$-\omega r a \delta^*_{i-1,j} - (1 + 2\omega r a)\delta^*_{ij} - \omega r a \delta^*_{i+1,j} = p_{ij} \tag{5.1}$$

for $i = 1, 2, \ldots, N-1$, where $r = k/h^2$ and p_{ij} is given by the right hand side of (4.5a). Note that $\nabla U^n_{ij} = \delta^{n-1}_{ij}$ which can be saved from the previous time step. Since in general this is a three level scheme, two time levels are required as initial data. In our numerical experiments we used the exact solution at time 0 and k, but in practice a two-level scheme must be used for the first step.

When Dirichlet conditions are imposed, the system (5.1) is completed by specifying δ^*_{0j} and δ^*_{Nj} as discussed in Section 4. For the Robbins condition we discretize the boundary conditions obtained for δ^* in Section 4. For example, when α and β are constant we discretize (4.16) as

$$\alpha \delta^*_{bj} + \frac{\beta}{2h}(\delta^*_{b+1,j} - \delta^*_{b-1,j}) = \hat{p}_{bj} \tag{5.2}$$

for $b = 0$ or N, with $\hat{p}_{bj}$ representing the right hand side of (4.16). This can be combined with equation (5.1) for $i = b$ to eliminate δ^*_{-1} (when $b = 0$) or δ^*_{N+1} (when $b = N$). For example, at the right boundary ($b = N$) we obtain

$$\delta^*_{N+1,j} = \delta^*_{N-1,j} + \frac{2h}{\beta}(\hat{p}_{Nj} - \alpha \delta^*_{Nj}) \tag{5.3}$$

from (5.2). Inserting this in (5.1) gives the equation

$$-2\omega r a \delta^*_{N-1,j} + [1 + 2\omega r a(1 + h\alpha/\beta)]\, \delta^*_{Nj} = p_{Nj} + \frac{2\omega r a h}{\beta}\hat{p}_{Nj}$$

with a similar equation obtained for $i = 0$. We thus have a tridiagonal system of $N + 1$ equations for δ^*_{ij}, $i = 0, 1, \ldots, N$.

Note that the expression for p_{ij} (the right hand side of (4.5a)) involves second differences of U^n_{ij} and δ^{n-1}_{ij} in the x-direction and hence evaluating p_{Nj} requires values $U^n_{N+1,j}$ and $\delta^n_{N+1,j}$. These can be obtained in the same way we derived (5.3) by discretizing (1.2b). We obtain

$$U^n_{N+1,j} = U^n_{N-1,j} + \frac{2h}{\beta}(\gamma^n_j - \alpha U^n_{Nj}) \tag{5.4}$$

and

$$\delta^{n-1}_{N+1,J} = U^n_{N+1,J} - U^{n-1}_{N+1,J}.$$

Actually, in practice we have found that the use of (5.4) leads to a considerable loss of accuracy near the boundary which can be avoided by using a higher order approximation to the derivtive. If (1.2b) is instead discretized by

$$\alpha U^n_{Nj} + \frac{1}{6h}\beta(2U^n_{N+1,J} + 3U^n_{Nj} - 6U^n_{N-1,J} + U^n_{N-2,J}) = \gamma^n_j$$

we obtain

$$U^n_{N+1,J} = \frac{1}{2}(-3U^n_{Nj} + 6U^n_{N-1,J} - U^n_{N-2,J}) + \frac{6h}{\beta}(\gamma^n_j - \alpha U^n_{Nj}) \tag{5.5}$$

in place of (5.4). This did not seem to adversely affect the numerical stability but improved the errors considerably.

When α depends of y and t (and $\beta = 1$ as discussed in Section 4), the boundary condition (4.17) is discretized to again give (5.3) with α replaced by α^{n+1}_j and

$$\hat{p}_{Nj} = (\gamma^{n+1}_j - \gamma^n_j) - (\alpha^{n+1}_j - \alpha^n_j)U^n_{Nj} - \omega k^2 c[\gamma^n_{yy} - \alpha^n_{yy}U^n_{Nj} - \alpha^n_{yy}\delta^{n-1}_{Nj}/k - \alpha^n_y(U^n_{N,J+1} - U^n_{N,J-1})/h$$
$$-\alpha^n_y(\delta^{n-1}_{N,J+1} - \delta^{n-1}_{N,J-1})/kh - \alpha^n_t(U^n_{N,J+1} - 2U^n_{Nj} + U^n_{N,J-1})/h^2].$$

6. Multi-step methods

In many situations it is necessary to use fractional step methods involving more than two steps. Consequently, additional intermediate solutions arise for which boundary conditions must also be specified.

For example, the LOD and ADI methods are easily extended to three space dimensions by adding a third step to the process. We consider the LOD method, which becomes

$$(I - \frac{1}{2}kD^2_x)U^* = (I + \frac{1}{2}kD^2_x)U^n \tag{6.1a}$$

$$(I - \frac{1}{2}kD^2_y)U^{**} = (I + \frac{1}{2}kD^2_y)U^* \tag{6.1b}$$

$$(I - \frac{1}{2}kD^2_z)U^{n+1} = (I + \frac{1}{2}kD^2_z)U^{**}. \tag{6.1c}$$

For clarity we have dropped the (three) subscripts on these variables and will also suppress the spatial

arguments of functions where convenient. The method (6.1) corresponds to the splitting

$$L = L^* + L^{**} + L^{***}$$

with L^* and L^{**} given by (1.6) and $L^{***} = \partial_z^2$. The boundary conditions for u^* along $x = x_b$ are again given by (2.3), but now we have

$$u_{xx} = (L - L^{**} - L^{***})u = u_t - u_{yy} - u_{zz}$$

$$u_{xxxx} = (L - L^{**} - L^{***})^2 u.$$

The boundary conditions for U^* along $x = x_b$ become

$$U^*_{bij} = [I - k(\partial_y^2 + \partial_z^2) + \frac{1}{2}k^2(\partial_y^2 + \partial_z^2)^2]\, u(x_b, y_i, z_j, t_n + k)\,. \tag{6.2}$$

Note that this involves only tangential derivatives along the boundary $x = x_b$.

In the second step, we need boundary data for U^{**} along $y = y_b$. Recall that we view U^{**} as an approximation to $u^{**}(t_n+2k)$, where u^{**} satisfies (1.4b) with initial conditions $u^{**} = u^*$ at $t = t_n+k$. Hence we can expand

$$u^{**}(t_n+2k) = [I + kL^{**} + \frac{1}{2}k^2(L^{**})^2 + \cdots]\, u^*(t_n+k). \tag{6.3}$$

Using the previously determined expansion (1.10) in (6.3) gives boundary data for U^{**} in terms of $u(t_n)$.

Actually, for three-step methods such as this, there is a much easier approach. By (6.1c), we can view U^{**} as an approximation to the function obtained by solving (1.4b) *backwards* in time from $U^{n+1} = u(t_{n+1})$ With time step $-k$ we thus obtain

$$U^{**} = [I - kL^{***} + \frac{1}{2}k^2(L^{***})^2 - \ldots]\, u(t_{n+1})$$

so that the boundary conditions are

$$U^{**}_{ibj} = u - ku_{zz} + \frac{1}{2}k^2 u_{zzzz}$$

all evaluated at (x_i, y_b, z_j, t_{n+1}).

Finally, we note that even in two space dimensions it may be necessary to use a three-step method if the operator L is split as the sum of two noncommuting operators, as is typically the case in variable coef-

ficient or nonlinear problems. Then second order accuracy is retained by using a three-step Strang splitting[12].

Table 2.1

BC	N	k	L_∞-error	L_2-error	L_2-rate
	11	.15	8.14E-3	3.69E-3	
(2.11)	21	.075	4.50E-3	1.97E-3	.91
	31	.05	3.17E-3	1.35E-3	.93
	11	.15	1.54E-3	7.05E-4	
(2.12)	21	.075	4.38E-4	1.94E-4	1.86
	31	.05	2.04E-4	8.93E-5	1.91
	11	.15	1.07E-4	5.49E-5	
(2.5)	21	.075	1.51E-5	8.66E-6	2.66
	31	.05	4.73E-6	3.04E-6	2.58

Table 2.2

BC	N	k	L_∞-error	L_2-error	L_2-rate
	11	.15	1.55E-2	5.28E-3	
(2.11)	21	.075	9.87E-3	3.02E-3	.81
	31	.05	7.23E-3	2.11E-3	.88
	11	.15	2.64E-3	1.06E-3	
(2.12)	21	.075	7.94E-4	2.93E-4	1.86
	31	.05	3.76E-4	1.34E-4	1.93
	11	.15	1.02E-4	4.45E-5	
(2.5)	21	.075	1.59E-5	8.59E-6	2.37
	31	.05	6.09E-6	3.41E-6	2.28

Table 4.1

BC	N	k	L_∞-error	L_2-error	L_2-rate
(4.10)	11	.15	2.12E-3	9.55E-4	
	21	.075	5.75E-4	2.50E-4	1.93
	31	.05	2.63E-4	1.13E-4	1.96
(4.9)	11	.15	8.65E-4	2.71E-4	
	21	.075	2.57E-4	7.80E-5	1.80
	31	.05	1.20E-4	3.64E-5	1.88

Table 4.2

BC	N	k	L_∞-error	L_2-error	L_2-rate
(4.20a)	11	.06	1.35E-3	4.38E-4	
	21	.03	4.11E-4	1.15E-4	1.93
	31	.02	1.94E-3	5.20E-5	1.96
(4.20b)	11	.06	6.55E-4	2.41E-4	
	21	.03	1.64E-4	5.57E-5	2.11
	31	.02	7.32E-5	2.41E-5	2.07
(4.20c)	11	.06	1.40E-4	4.96E-5	
	21	.03	3.46E-4	9.80E-6	2.34
	31	.02	1.54E-5	4.00E-6	2.21

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16. Abstract

A general approach to determining the correct intermediate boundary conditions for dimensional splitting methods is presented and illustrated. The intermediate solution U^* is viewed as a second-order accurate approximation to a modified equation. Deriving the modified equation and using the relationship between this equation and the original equation allows us to determine the correct boundary conditions for U^*. To illustrate this technique, we apply it to LOD and ADI methods for the heat equation in two and three space dimensions. The approximate factorization method is considered in slightly more generality.

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