Nonlinear Hyperbolic Problems: Theoretical, Applied, and Computational Aspects

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A Linear Hyperbolic System with Stiff Source Terms

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SUMMARY

We consider the behavior of solutions to the system $u_t+v_x=0$, $v_t+u_x=(\alpha u-v)/\tau$ as the relaxation parameter τ approaches 0. If the subcharacteristic condition $|\alpha|\leq 1$ is satisfied then u approaches a solution to the equilibrium equation $u_t+\alpha u_x=0$ and v approaches αu. If this condition is violated then solutions blow up as $\tau\to 0$. If viscous terms are added to the right hand side of this system, then the behavior depends on the relative rates at which the viscous parameter and relaxation time vanish. With sufficient viscosity, convergence to the equlibrium solution is again observed.

1. Introduction

Consider the linear hyperbolic system

$$\begin{aligned} u_t+v_x &= 0 \\ v_t+u_x &= (\alpha u-v)/\tau. \end{aligned} \tag{1}$$

This is a very simple linearized model for propagation with relaxation as occurs in a relaxing gas, for example. The corresponding homogeneous system is equivalent to the linear wave equation and has *frozen propagation speeds* of ± 1. The nonhomogeneous term in the second equation tends to drive v towards the equilibrium value $v=\alpha u$ with relaxation time scale τ. If we consider the reduced equation obtained by eliminating the second equation and replacing v by αu in the first equation, we obtain the advection equation

$$u_t+\alpha u_x=0 \tag{2}$$

with the *equilibrium propagation speed* α.

We study the behavior of the solution to (1) in the singular perturbation limit $\tau\to 0$. More complicated nonlinear equations of this same general form have been studied for some time in a variety of contexts, and linearized equations with a form similar to (1) are often used to determine stability (for example, see the discussion of flood waves in Chapter 3 of Whitham[6] or the discussion of a relaxing gas by Clarke[3]). For the system (1), stability requires $|\alpha|\leq 1$. This is called the *subcharacteristic condition* since this condition guarantees that the characteristic speed of the equilibrium equation (2) lies between the characteristic speeds of the full system. In this case the equilibrium equation determines the asymptotic behavior as $\tau\to 0$.

Here we are concerned primarily with the case $|\alpha|>1$ and our investigation was motivated by curiosity about the behavior of the solution in this case. Now the equilibrium equation cannot be the correct limiting equation as $\tau \to 0$ since the hyperbolic system (1) allows propagation with speed 1 at most. Instead, the solution blows up as $\tau \to 0$ along the characteristic $x = \mathrm{sgn}(\alpha)t$.

We also consider what happens if dissipative terms are added to the right hand side of (1), obtaining

$$\begin{aligned} u_t + v_x &= \epsilon u_{xx} \\ v_t + u_x &= (\alpha u - v)/\tau + \epsilon v_{xx}. \end{aligned} \tag{3}$$

This parabolic system allows infinite propagation speeds and so it is again possible that as $\tau \to 0$ propagation with speed α will be observed, even for $|\alpha| > 1$.

The behavior depends on the relation between ϵ and τ. If we set

$$\epsilon = \delta\tau \tag{4}$$

with δ fixed, so that $\epsilon = O(\tau)$ as $\tau \to 0$, then we find that the behavior depends on the magnitude of δ. If $\delta < \alpha^2 - 1$ then blowup can still occur, while for $\delta > \alpha^2 - 1$ convergence to a solution of the equilibrium equation is observed.

Here we consider only the Riemann problem for this equation with data

$$u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases} \quad \text{and} \quad v(x,0) = \alpha u(x,0). \tag{5}$$

Behavior similar to that illustrated here has also been observed with other initial data.

2. Equivalent forms

Note that the parameter τ can be eliminated by setting

$$u(x,t) = \bar{u}(x/\tau,\, t/\tau), \quad v(x,t) = \bar{v}(x/\tau,\, t/\tau). \tag{6}$$

Then $(\bar{u}, \bar{v})$ satisfies equation (1) with $\tau = 1$,

$$\begin{aligned} \bar{u}_t + \bar{v}_x &= 0 \\ \bar{v}_t + \bar{u}_x &= \alpha\bar{u} - \bar{v} \end{aligned} \tag{7}$$

We study the behavior of the solution (u, v) at fixed time $t = 1$ as $\tau \to 0$ or, equivalently, the long-time behavior of $(\bar{u}, \bar{v})$ with $\tau = 1$ and $t \to \infty$.

Also note that if we differentiate the first equation of (1) with respect to t, the second equation with respect to x, and then combine them to eliminate v, we obtain a second order scalar equation for u,

$$\tau(u_{tt} - u_{xx}) + (u_t + \alpha u_x) = 0 \tag{8}$$

with initial data $u(x,0) = u_0(x)$ and $u_t(x,0) = -v_0'(x)$. The equilibrium equation (2) is clearly a singular perturbation limit of this equation as $\tau \to 0$. The same equation is obtained for v (though with different initial data) in a similar way.

For the Riemann problem it is possible to compute the exact solution to (8) using Riemann functions (see, for example, [4], [5]), obtaining

$$\begin{aligned} u(x,t) = {} & \exp\left(\frac{\alpha x - t}{2\tau}\right)\left\{\frac{1}{2}\left((1+\alpha)\,u_l + (1-\alpha)\,u_r\right) J_0\left(2\sqrt{\lambda(x^2 - t^2)}\right)\right. \\ & + u_r\left(\frac{1-\alpha}{4\tau}\right)\int_0^{x+t} J_0\left(2\sqrt{\lambda(x-t)(x+t-\xi)}\right)\exp\left(\left(\frac{1-\alpha}{4\tau}\right)\xi\right)\,d\xi \\ & \left. + u_l\left(\frac{1+\alpha}{4\tau}\right)\int_{x-t}^{0} J_0\left(2\sqrt{\lambda(x+t)(x-t-\xi)}\right)\exp\left(-\left(\frac{1+\alpha}{4\tau}\right)\xi\right)\,d\xi\right\} \end{aligned} \tag{9}$$

where $\lambda = (1-\alpha^2)/16\tau^2$ and J_0 is the Bessel function of order 0. As it stands, this form of the solution is not very illuminating, although asymptotic expansions can be performed to derive some of the results presented below.

The characteristic form of (1) will also prove useful. Let r and s be the characteristic variables

$$r = u + v, \qquad s = u - v.$$

Then

$$\begin{aligned} r_t + r_x &= \psi(r,s) \\ s_t - s_x &= -\psi(r,s), \end{aligned} \tag{10}$$

with
$$\psi(r,s) = \frac{1}{2\tau}\left((\alpha - 1)\,r + (\alpha+1)\,s\right).$$

3. The subcharacteristic case

If $\alpha = \pm 1$, then for the Riemann problem (5) one of the characteristic variables r or s is identically zero. The remaining characteristic equation from (10) reduces to the advection equation (2) and so the solution for any $\tau > 0$ agrees with the equilibrium solution.

If $|\alpha| < 1$ then the solution at fixed time t (say $t = 1$) behaves as shown on the left side of Figure 1 as $\tau \to 0$. The initial discontinuity in the Riemann data results in discontinuities along the characteristics at $x = \pm 1$, but the strength of these discontinuities decays exponentially as $\tau \to 0$. The magnitude of the jumps is easily calculated from the characteristic equations (10). Along the characteristic $x = t$, for example, the leftgoing characteristic variable s is identically equal to $s_r = (1-\alpha)u_r$. The rightgoing characteristic is equal to $r_r = (1+\alpha)u_r$ to the right of the characteristic line, at $x = t+$, but to the left of this line r evolves according to

$$r_t + r_x = \left(\frac{\alpha - 1}{2\tau}\right) r + \left(\frac{1-\alpha^2}{2\tau}\right) u_r.$$

This reduces to an ODE along $x = t-$, with initial data $r_l = (1+\alpha)u_l$. The solution is thus

$$r(t-, t) = (1+\alpha)\left(u_r + (u_l - u_r)e^{-(1-\alpha)t/2\tau}\right).$$

Across the rightgoing characteristic, the jumps in u and v are thus determined to be

$$[u] = [v] = \frac{1}{2}(1+\alpha)(u_r - u_l)e^{-(1-\alpha)t/2\tau}. \tag{11}$$

Similarly, across the leftgoing characteristic $x = -t$, the jumps are found to be

$$[u] = -[v] = \frac{1}{2}(1-\alpha)(u_r - u_l)e^{-(1+\alpha)t/2\tau}. \tag{12}$$

Note that these jumps decay exponentially for $|\alpha| < 1$. In between the characteristics, the solution (u, v) is smooth, and for fixed $t = 1$ approaches a discontinuity at $x = \alpha$ as $\tau \to 0$, as seen in Figure 1. The smooth profile is found to have width $\sqrt{\tau}$ as $\tau \to 0$.

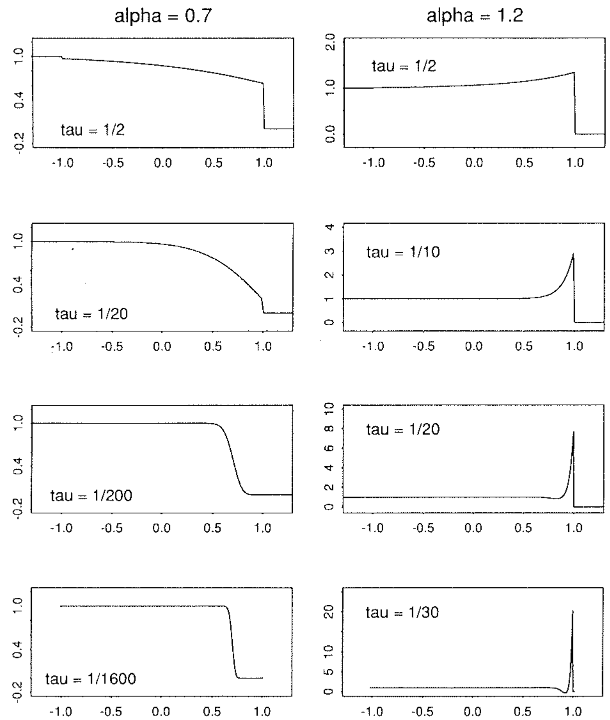

FIG. 1. *Behavior of the solution of (1) as* $\tau \to 0$ *for* $\alpha = 0.7$ *and* $\alpha = 1.2$.

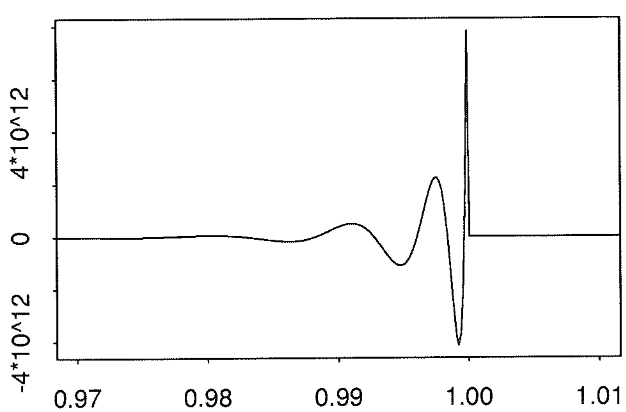

FIG. 2. *Behavior of the solution of (1) for* $\alpha = 1.2$ *and* $\tau = 1/300$.

Alternatively, in the scaled equation (7), the width of the transition is $\sqrt{t}$ as $t \to \infty$. This suggests taking an Ansatz of the form

$$\bar{u}(x,t) = U\left(\frac{x-\alpha t}{\sqrt{t}}, t\right) \equiv U(z,t). \tag{13}$$

Equation (7) can then be written in the form

$$\begin{aligned}(1-\alpha^2)U_{zz} + \tfrac{1}{2}zU_z &= t(U_{tt}+U_t) + (2\alpha t^{1/2}+z)U_{tz} \\ &\quad + \left(\alpha t^{-1/2}z + \tfrac{1}{4}t^{-1}z^2\right)U_{zz} + \tfrac{3}{4}t^{-1}zU_z.\end{aligned} \tag{14}$$

If we assume that $\bar{u}(x,t)$ approaches a function of z alone for large t, then this asymptotic form $U(z)$ can be determined by setting the right hand side of (14) to zero (the time derivative terms vanish while the terms involving negative powers of t also vanish as $t \to \infty$). The asymptotic form thus satisfies the ODE

$$U_{zz} = \frac{-z}{2(1-\alpha^2)}U_z.$$

The solution has the form

$$U(z) = u_l + (u_r - u_l)\,\mathrm{erf}\left(\frac{z}{2\sqrt{1-\alpha^2}}\right)$$

which gives the asymptotic form

$$\bar{u}(x,t) \sim u_l + (u_r - u_l)\,\mathrm{erf}\left(\frac{x-\alpha t}{2\sqrt{(1-\alpha^2)t}}\right).$$

Note that this is the solution of the convection-diffusion equation

$$\bar{u}_t + \alpha\bar{u}_x = (1-\alpha^2)\bar{u}_{xx}. \tag{15}$$

Another way to see that this is the expected behavior is to set

$$\bar{v}(x,t) = \alpha\bar{u}(x,t) + \bar{v}_1(x,t) \tag{16}$$

and to assume, following Chen and Liu[2], that $\bar{v}_1$ is small and its derivatives even smaller. From the second equation of (7) we obtain

$$\bar{v}_1 = -(\bar{v}_t + \bar{u}_x) \approx (\alpha^2 - 1)\bar{u}_x.$$

Using (16) with this expression for $\bar{v}_1$ in the first equation of (7) then gives the convection-diffusion equation (15).

4. The supercharacteristic case

If $|\alpha| > 1$ then there is exponential blowup of the solution at the characteristic $x = \mathrm{sgn}(\alpha)t$, as seen from (11) or (12). Figure 1 shows some typical results. Note the rescaling of the vertical axis as τ decreases.

These results were calculated by a numerical method based on the characteristic form (10) with $\Delta t = \Delta x$. We integrate the resulting coupled set of ODEs along the characteristics using the trapezoidal rule, obtaining the point-implicit method

$$R_j^{n+1} = R_{j-1}^n + \frac{\Delta t}{2}[\psi(R_{j-1}^n, S_{j-1}^n) + \psi(R_j^{n+1}, S_j^{n+1})]$$

$$S_j^{n+1} = S_{j+1}^n - \frac{\Delta t}{2}[\psi(R_{j+1}^n, S_{j+1}^n) + \psi(R_j^{n+1}, S_j^{n+1})].$$

By using this characteristic method we insure that the discontinuity in our numerical solution remains perfectly sharp and that numerical information cannot propagate at speeds greater than 1.

Figure 2 shows an expanded view of the solution with a smaller value of τ. The oscillations seen in Figure 2 are not numerical artifacts; the solutions are in fact highly accurate. To investigate the form of the solution near the characteristic along which blowup occurs, we consider the case $\alpha > 1$ for concreteness. We then wish to investigate the behavior of the solution near $x = t$. The asymptotic form can be rigorously determined from the exact solution (9), but here we present a more illuminating derivation.

Set

$$\bar{u}(x,t) = w(y,t) \quad \text{with } y = x - t \le 0.$$

Then

$$w_{tt} - 2w_{yt} + w_t + (\alpha - 1)w_y = 0. \tag{17}$$

We know the solution grows exponentially with rate $e^{\beta t}$ with $\beta = (\alpha-1)/2$, and it is found that the proper scaling of the solution as t varies is obtained by looking for a solution of the form

$$w(y,t) \sim W(\eta)e^{\beta t}$$

where $\eta = ty$. The equation (17) then reduces to the ODE

$$(y^2 - 2\eta)W''(\eta) + ((2\beta+1)y - 2)W'(\eta) + (\beta^2 + \beta)W(\eta) = 0.$$

If we fix η and let $t \to \infty$, so $y = \eta/t \to 0$, we obtain

$$-2\eta W''(\eta) - 2W'(\eta) + \frac{1}{4}(\alpha^2 - 1)W(\eta) = 0.$$

By setting

$$W(\eta) = Z(\sigma) \quad \text{with } \sigma = \sqrt{-\eta\left(\frac{\alpha^2-1}{2}\right)} \quad \text{for } \eta \le 0,$$

this can be reduced to

$$Z''(\sigma) + \sigma^{-1}Z'(\sigma) + Z(\sigma) = 0,$$

which is the equation for the Bessel function J_0, so that $W(\eta) = J_0(\sigma)$. Unraveling the various changes of variables, we finally obtain

$$u(x,t) \sim J_0\left(\sqrt{-t(x-t)\left(\frac{\alpha^2-1}{2\tau^2}\right)}\right) e^{(\alpha-1)t/2\tau}$$

for $x \sim t-$. The Bessel function gives rise to the oscillations seen in Figure 2.

5. The viscous regularization

Consider now the viscous equations

$$\begin{aligned} u_t + v_x &= \epsilon u_{xx} \\ v_t + u_x &= (\alpha u - v)/\tau + \epsilon v_{xx} \end{aligned} \tag{18}$$

We wish to consider limits as $\tau \to 0$ and $\epsilon \to 0$. Here we consider the case $\epsilon = \delta\tau$ with δ fixed. Setting $u(x,t) = \bar{u}(x/\tau, t/\tau)$ as before then leads to the system

$$\begin{aligned} \bar{u}_t + \bar{v}_x &= \delta \bar{u}_{xx} \\ \bar{v}_t + \bar{u}_x &= \alpha\bar{u} - \bar{v} + \delta\bar{v}_{xx} \,. \end{aligned} \tag{19}$$

We are thus interested in the long-time behavior of this system with δ fixed.

If we again assume that (16) holds with v_1 small, then the second equation of (19) gives

$$\bar{v}_1 \approx (\alpha^2 - 1)\bar{u}_x + \delta\bar{v}_{xx}.$$

Using this in the first equation of (19), along with $\bar{v} \approx \alpha\bar{u}$, yields the equation

$$\bar{u}_t + \alpha\bar{u}_x \approx (\delta - (\alpha^2 - 1))\bar{u}_{xx} - \alpha\delta\bar{u}_{xxx}. \tag{20}$$

This equation is well-posed only if

$$\delta > \alpha^2 - 1.$$

Otherwise, we expect exponential growth in the solution.

In fact, we can see more rigorously that $\delta = \alpha^2 - 1$ is the correct cutoff for exponential growth by considering the Fourier transform and observing that there are exponentially growing modes precisely when $\delta < \alpha^2 - 1$. Fourier transforming (19) leads to

$$\frac{\partial}{\partial t}\begin{bmatrix} \hat{u}(\xi,t) \\ \hat{v}(\xi,t) \end{bmatrix} = G(\xi)\begin{bmatrix} \hat{u}(\xi,t) \\ \hat{v}(\xi,t) \end{bmatrix}$$

where the matrix $G(\xi)$ is given by

$$G(\xi) = -\begin{bmatrix} \delta\xi^2 & i\xi \\ i\xi - \alpha & 1 + \delta\xi^2 \end{bmatrix}.$$

Exponential growth occurs if any of the eigenvalues of $G(\xi)$ have positive real part. The eigenvalues are

$$\lambda(\xi) = -\frac{1}{2}\left(1 + 2\delta\xi^2 \pm \sqrt{1 - 4\xi^2 + 4i\alpha\xi}\right)$$

and the one with larger real part has

$$\mathrm{Re}(\lambda(\xi)) = -\xi^2\,\mathrm{Re}\left(\frac{\sqrt{1 - 4\xi^2 + 4i\alpha\xi} - 1}{2\xi^2} - \delta\right).$$

Since

$$\max_{\xi}\,\mathrm{Re}\left(\frac{\sqrt{1 - 4\xi^2 + 4i\alpha\xi} - 1}{2\xi^2}\right) = \alpha^2 - 1$$

(attained at $\xi = 0$), the eigenvalues all have nonpositive real part provided $\delta > \alpha^2 - 1$.

Figure 3 shows some typical solutions for $\alpha = 1.2$, in which case $\alpha^2 - 1 = 0.44$. For $\delta = 0.35$ the solution grows and also becomes more oscillatory as $\tau \to 0$. For $\delta = 0.45$

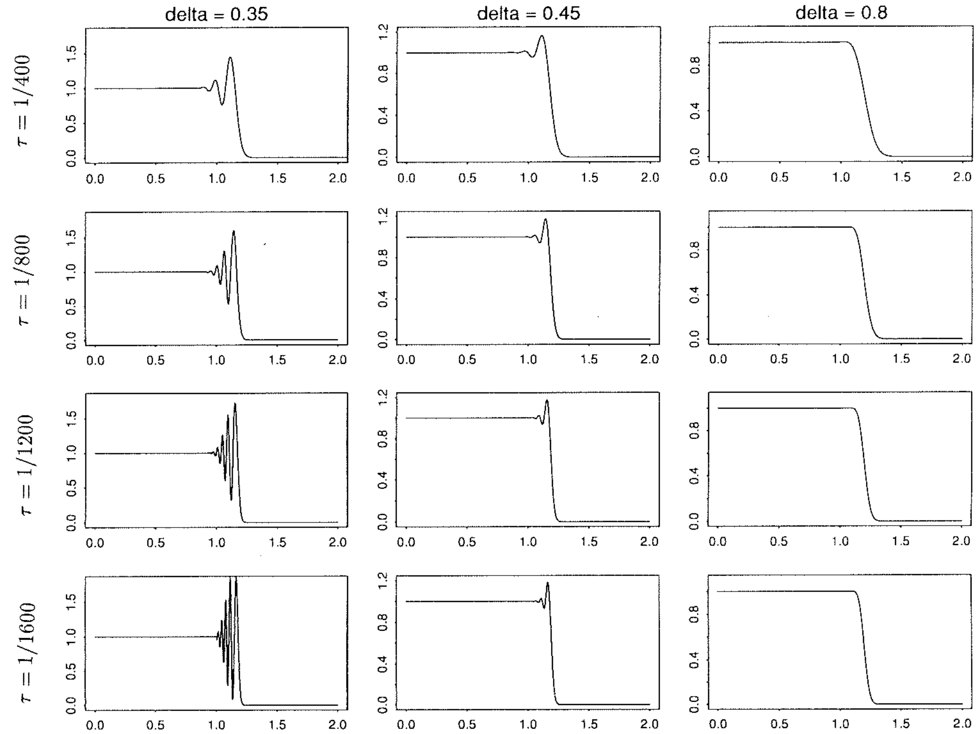

FIG. 3. *Behavior of the solution to the viscous system (18) as* $\tau \to 0$ *for* $\alpha = 1.2$ *and various values of* δ.

the solution is oscillatory but the shape remains fixed as $\tau \to 0$ while the width of the transition region shrinks with τ, so that pointwise convergence to the solution of the equilibrium equation is observed (which with $t = 1$ is simply a discontinuity at $x = 1.2$). For $\delta = 0.8$ there is sufficient viscosity that the solution is no longer oscillatory.

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