Proceedings of Symposia in Applied Mathematics

Two-layer shallow water system and its applications

Jihwan Kim and Randall J. LeVeque

Abstract. The multi-layer shallow water system is derived by depth averaging the incompressible Navier-Stokes equations with the hydrostatic assumption within layers. While the single layer shallow water system is hyperbolic, the two-layer system is conditionally hyperbolic because of the coupling terms between the layers. The eigenstructure of the system cannot be found in closed form and the eigenvalues may become imaginary number. In this work, we assume that the system conserves hyperbolicity. The eigenvalues are computed numerically, and the f-wave approach is used to balance the source term. As an application, we consider a tsunami generated by an underwater landslide.

1. Introduction

The two-layer shallow water system is given as

$$
\begin{aligned}
(h_1)_t + (h_1u_1)_x &= 0, \\
(h_1u_1)_t + \left(h_1u_1^2 + \frac{1}{2}gh_1^2\right)_x &= -gh_1(h_2)_x - gh_1b_x, \\
(h_2)_t + (h_2u_2)_x &= 0, \\
(h_2u_2)_t + \left(h_2u_2^2 + \frac{1}{2}gh_2^2\right)_x &= -\rho_0gh_2(h_1)_x - gh_2b_x,
\end{aligned}
\tag{1.1}
$$

where h_1 and u_1 denotes the depth and velocity of the upper layer, and h_2 and u_2 correspond to the lower layer. And $b(x)$ is the bottom topography, ρ_0 is the ratio of the densities($\rho_0 = \rho_1/\rho_2 < 1$), and g is the gravity constant.

The first and third equations indicate the conservation of mass, and the second and fourth equations state the conservation of the momentum for each layer. The system contains non-conservative products by which the momentum of two layers are coupled.

When we solve the two-layer system, several difficulties arise. First, the system is conditionally hyperbolic. If the difference of the velocities of two layers becomes large enough, then the system loses hyperbolicity, and we expect Kelvin-Helmholtz instability. Secondly, we cannot find the explicit expression for the eigenvalues of

Key words and phrases. Multi-layer shallow water equation, Finite volume method, Submarine landslide, f-wave method.

the two layer system. Also we need a numerical scheme which is well-balancing with source term since the system is non-conserve.

The two-layer shallow water system is accepted as numerical model not only for the flows with different densities but also for the tsunamis generated by underwater landslides. Castro et al. [**Castroetal:2007**], for example, applied this system to the strait of Gibraltar which is connecting the Atlantic Ocean and the Mediterranean Sea. As for the modeling of tsunamis generated by underwater landslides, Ostapenko [**Ostapenko:99b**], and Fernandez-Nieto et al. [**Fernandezetal:2008**] , for instances, used two-layer shallow water system considering underwater landslides as a viscous fluid.

2. Numerical Modeling

We can write in the following form,

$$q(x,t)_t + f(q)_x = S(q)$$

where

$$q(x,t) = \begin{bmatrix} h_1 \\ h_1u_1 \\ h_2 \\ h_2u_2 \end{bmatrix}, \quad f(q) = \begin{bmatrix} h_1u_1 \\ h_1u_1^2 + \frac{1}{2}gh_1^2 \\ h_2u_2 \\ h_2u_2^2 + \frac{1}{2}gh_2^2 \end{bmatrix},$$

and the source term $S(q)$ contains the bottom bathymetry. The source term is expressed as

$$S(q) = \begin{bmatrix} 0 \\ -gh_1(h_2)_x - gh_1b_x \\ 0 \\ -\rho_0gh_2(h_1)_x - gh_2b_x \end{bmatrix}.$$

This source term is difficult to handle since we expect not only smooth solutions but also shock wave solutions whose derivative is delta function.

In quasi-linear form the multi-layer SW system is written as

$$q_t + A(q)q_t = \psi(q),$$

where the Jacobian matrix $f'(q)$ is

$$A(q) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ gh_1 - u_1^2 & 2u_1 & gh_1 & 0 \\ 0 & 0 & 0 & 1 \\ \rho_0gh_2 & 0 & gh_2 - u_2^2 & 2u_2 \end{bmatrix}, \psi(q) = \begin{bmatrix} 0 \\ -gh_1b_x \\ 0 \\ -gh_2b_x \end{bmatrix}.$$

2.1. Eigenstructure.

The characteristic polynomial of this system is given by,

$$\left((\lambda - u_1)^2 - gh_1\right)\left((\lambda - u_2)^2 - gh_2\right) = \rho_0g^2h_1h_2.$$

The closed form of the eigenvalues can not be found in general case. Interesting geometrical interpretation is suggested in Ovsyannikov [**Ovsyannikov:1979**]. We introduce two variables p and q which are defined by

$$\lambda - u_1 = p\sqrt{gh_1}, \quad \lambda - u_2 = q\sqrt{gh_2}.$$

Then p and q satisfy

$$(p^2 - 1)(q^2 - 1) = \rho_0, \tag{2.1}$$

$$q = \sqrt{h_1/h_2}p + (u_1 - u_2)/\sqrt{gh_2}. \tag{2.2}$$

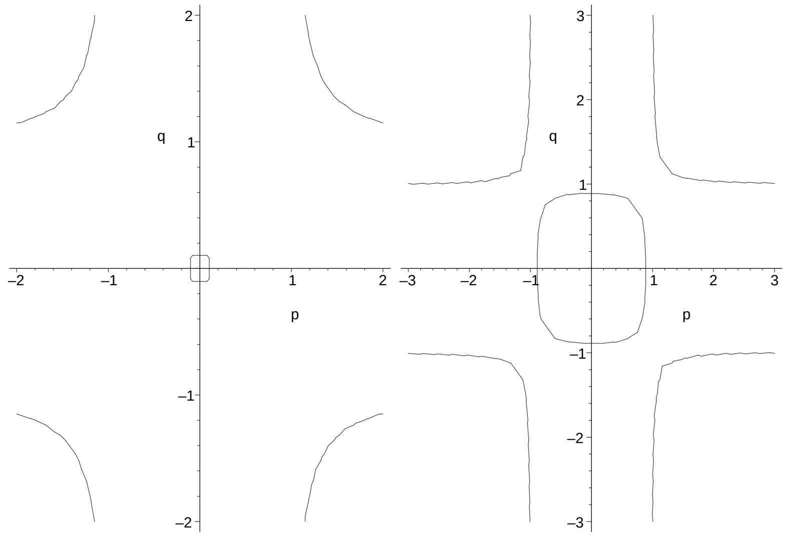

FIGURE 1. $\rho_0 = 0.98$ and $\rho_0 = 0.2$

The Figure 1 shows the curves (2.1) for different values of ρ_0 which has two components. The inside component is circular shape with radius of $\sqrt{1-\rho_0}$. Thus the center curve has smaller radius as the density ratio approaches to 1. The curves (2.1) and the line (2.2) have at least two intersections, and maximum of four. When there is no intersection with the central curve, the system has only two real eigenvalues when the system is non-hyperbolic.

2.1.1. *The Eigenvalues.* As we have mentioned, the explicit expression for the eigenvalues cannot be found. Instead we seek for the approximation in order to analyze the system. Let $f(\lambda)$ be the characteristic polynomial, that is,

$$f(\lambda) = \left((\lambda - u_1)^2 - gh_1\right)\left((\lambda - u_2)^2 - gh_2\right) - \rho_0 g^2 h_1 h_2.$$

Then we can find that

$$f'((u_1+u_2)/2) = 0, \quad f(u_1 \pm \sqrt{gh_1}) < 0, \quad \text{and} \quad f(u_2 \pm \sqrt{gh_2}) < 0.$$

Thus the hyperbolicity condition is $f((u_1+u_2)/2) > 0$.

$$f((u_1+u_2)/2) = \left(((u_2-u_1)/2)^2 - gh_1\right)\left(((u_1-u_2)/2)^2 - gh_2\right) - \rho_0 g^2 h_1 h_2 > 0.$$

We can see that the hyperbolicity condition depends on the value $(u_1 - u_2)^2$ and the system remains hyperbolic when that value is small.

We can find the hyperbolicity condition with some assumptions. If $\mid u_2 - u_1 \mid$ and $(1-r)$ are small, we can find approximate expressions for the eigenvalues. There are two eigenvalues which are always real,

$$\lambda^{\pm}_{ext} = U_m \pm \sqrt{g(h_1+h_2)}.$$

And the other two eigenvalues are conditionally real,

$$\lambda^{\pm}_{int} = U_c \pm \sqrt{g' \frac{h_1 h_2}{h_1+h_2}\left[1 - \frac{(u_2-u_1)^2}{g'(h_1+h_2)}\right]},$$

where

$$U_m = \frac{h_1 u_1 + h_2 u_2}{h_1 + h_2}, \quad U_c = \frac{h_1 u_2 + h_2 u_1}{h_1 + h_2},$$

and $g' = (1 - \rho_0)g$ is the reduced gravity. For this approximation the two-layer shallow water system is conditionally hyperbolic if

$$(u_1 - u_2)^2 < g'(h_1 + h_2),$$

and this condition is linked with the Kelvin-Helmholtz instability of the stratified flows.

The conditions for the hyperbolicity of the system can analytically be found for special cases. When $h_1 = h_2 = h$, the characteristic polynomial is quadratic. If the condition is satisfied

$$2\left(gh(1 - \sqrt{\rho_0})\right)^{1/2} < (u_2 - u_1)^2 < 2\left(gh(1 + \sqrt{\rho_0})\right)^{1/2},$$

the problem is ill-posed.

When $\rho_0 = 1$, the problem is ill-posed if

$$0 < (u_2 - u_1)^2 < \left((gh_1)^{1/3} + (gh_2)^{1/3}\right)^3,$$

which indicates that it is ill-posed if the speeds of two layers are different.

2.1.2. *Eigenvector.* Once the eigenvalues are found, the eigenvectors are given by

$$r^{(k)} = \begin{pmatrix} 1 \\ \lambda^{(k)} \\ c^{(k)} \\ c^{(k)}\lambda^{(k)} \end{pmatrix},$$

where $c^{(k)}$ is a constant,

$$c^{(k)} = -1 + \frac{(\lambda^{(k)} - u_1)^2}{gh_1}.$$

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3. Modeling Submarine Landslides

3.1. Friction Forces. One of the approaches in modeling underwater landslide is is to treat it as incompressible liquid. Physically it is natural to introduce friction forces between layers and bottom. For turbulent flow, the friction loss is usually approximated by the Manning equation,

$$f = -\frac{n^2 u|u|}{h^{1/3}},$$

where n is the Manning coefficient.

For our application, we can rewrite our system with friction as follows,

$$\begin{aligned} (h_1)_t + (h_1 u_1)_x &= 0, \\ (h_1 u_1)_t + \left(h_1 u_1^2 + \frac{1}{2} g h_1^2\right)_x &= -gh_1(h_2 + b)_x + f_1, \\ (h_2)_t + (h_2 u_2)_x &= 0, \\ (h_2 u_2)_t + \left(h_2 u_2^2 + \frac{1}{2} g h_2^2\right)_x &= -gh_2(\rho_0 h_1 + b)_x + f_2 - \rho_0 f_1 + f_{dry}, \end{aligned} \tag{3.1}$$

where

$$f_1 = -\frac{n_1^2 g(u_1 - u_2)|u_1 - u_2|}{h_1^{1/3}}, \tag{3.2}$$

$$f_2 = -\frac{n_2^2 g u_2 |u_2|}{h_2^{1/3}}. \tag{3.3}$$

In this expression n_1 and n_2 are Manning coefficients. The friction term f_1 is introduced to represent the friction between two layers, and f_2 denotes the friction between flow and bottom.

3.2. Internal Friction. We introduce f_{dry} to represent the internal friction in the landslide. Ostapenko [**Ostapenko:99b**], for instance, introduced "dry friction" as follows,

$$f_{dry} = \begin{cases} -\gamma\theta P u_2/|u_2|, & u_2 \neq 0, \\ \theta P \Phi/|\Phi|, & u_2 = 0, \quad |\Phi| > \theta P, \\ \Phi, & u_2 = 0, \quad |\Phi| < \theta P, \end{cases}$$

where

$$P = \frac{\rho_1 g h_1 + \rho_2 g h_2}{\rho_2} = g(\rho_0 h_1 + h_2).$$

Here P is the specific pressure of the landslide on the bottom, $\Phi = gh_2(\rho_0 h_1 + b)_x$ is the force acting on unit mass of the landslide in the horizontal direction,

$$\gamma = \frac{1}{1 + a|u_2|^2},$$

with a is a positive constant, $\theta = \tan\beta$ is the factor of "dry friction" of the landslide on the bottom, and β is the so-called angle of internal friction in the landslide.

The positive constant a is to be chosen to describe the motion of submarine landslides. As a approaches to ∞, γ goes to 0 which indicates that there is no f_{dry} friction between landslide and bottom. Similarly smaller a denotes that there are larger friction.

4. Numerical Tools

We use the finite volume method to solve the two-layer shallow water system. Since this system is non-conservetive, we need a numerical scheme which balances fluxes and source terms. Another problem arises from the fact that this system is conditionally hyperbolic. Lastly, we need a scheme that controls wet/dry interfaces and preserves depth positivity.

4.1. *f-wave* method. When we deal with the source terms, the fractional step method exhibits problems near steady-state. Because there is non-trivial balance between fluxes and source terms. One of the approaches dealing with the balance law is the *f-wave* method suggested by Bale et al. For details, we refer to [**BLMR:2002**]. The *f-waves* are based on deviation from steady state, and it is useful for problems with spatially varying flux function and source terms.

4.2. Preserving Hyperbolicity. The two-layer shallow water system is conditionally hyperbolic. Although the condition for the hyperbolicity can not be found explicitly, we can find it approximately when $|u_1 - u_2|$ and ρ_0 are small. This system is hyperbolic if

$$(u_1 - u_2)^2 < g'(h_1 + h_2),$$

where $g' = (1 - \rho_0)g$ is the reduced gravity. From this inequality, we can observe that the system is non-hyperbolic if the difference of the speeds of both layers becomes large. By introducing the friction forces between layers, we can ease the condition for hyperbolicity since the speeds are reduced.

4.3. Wet/Dry Interface. For the single layer shallow water system, one of our interests is in the motion of tsunamis near the shorelines where handling the wet/dry interfaces becomes crucial. To deal with this situation, we are required to find the height for the next time step, and to limit source terms. We refer to, for example, George[**George:2008**]. In two layer shallow water systems, the wet/dry interface problems arise not only near shorelines but also over variable topography. In order to develop numerical scheme for these cases, the basic property of the solver is to preserve steady state.

5. Numerical Tests

5.1. Test 1. For the first test, we consider fluids without friction terms. We picked this example to see the motion of the internal waves. Here the ratio of densities(ρ_0) is set as 0.6. The figure (5.1) demonstrates the speed of upper layer, interfaces and surface of fluids, and the speed of lower layer from top to bottom.

The initial conditions are given as the first figure in (5.1), and we can see the motion of internal waves as we expected.

5.2. Test 2. We use viscous model for this numerical test. For the first numerical test, the Manning coefficients are set as $n_1 = n_2 = 0.2$. The density ratio is chosen as $\rho_0 = 0.3$, and the critical angle β is picked as 0.3. The initial conditions are set as $u_1 = u_2 = 0$. The figures (5.2) show the motion of submarine landslides, the figure for $t = 5$ shows the steady state.

6. Conclusion and Future Works

We have shown well-balanced scheme for multi-layer shallow water system. And the numerical tests were robust as well. In this scheme, we find the eigenvalues and eigenvectors numerically based on Roe average. We presented a strategy for dealing with wet/dry interfaces which preserves steady state.

For the future works, we can modify Bingham fluid to represent the motion of landslide. Also we can extend this scheme for 2-dimensional model. On that case, developing adaptive meshed refinement can make the solver more efficient. Since the analytical solution cannot be found for this system, we are required to compare numerical results with the physical experiments.

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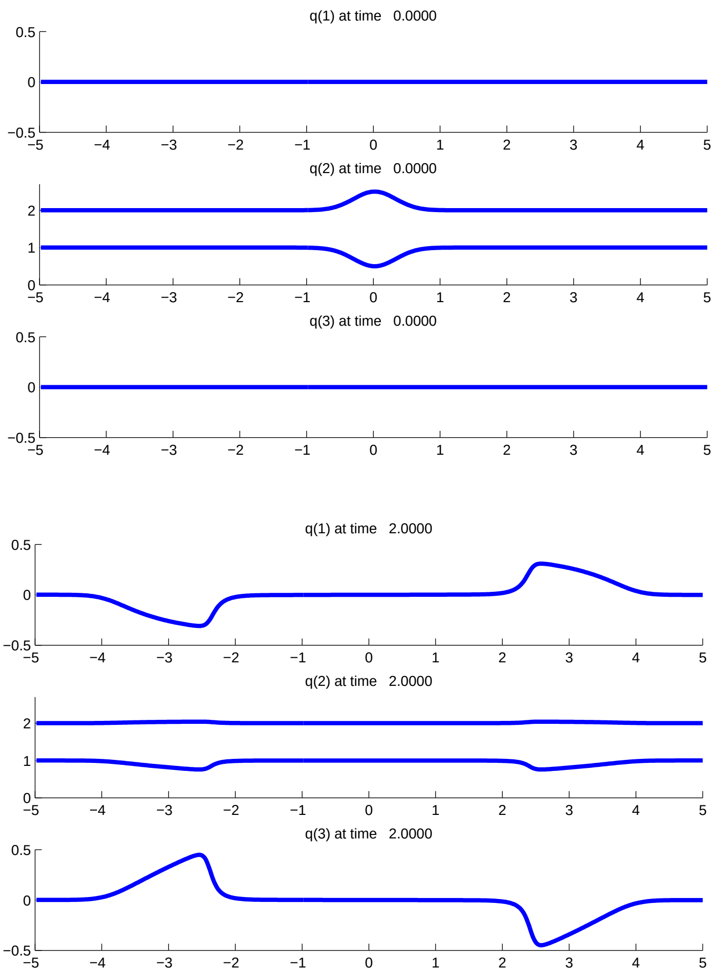

FIGURE 2. Test 1 for time = 0 and 2.

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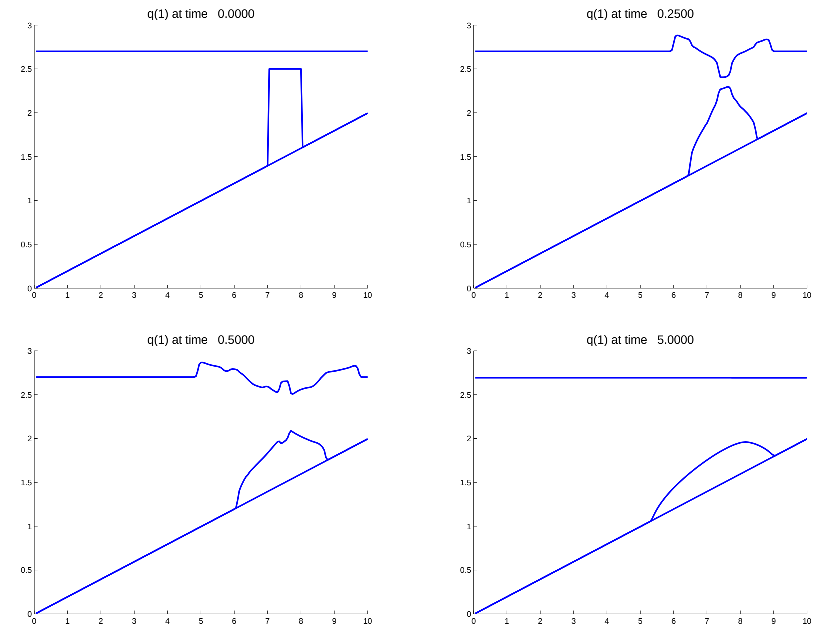

FIGURE 3. Test 2 for time $= 0, 0.25, 0.5$, and 5.

DEPARTMENT OF APPLIED MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98195

Current address: Department of Applied Mathematics, University of Washington, Seattle, Washington 98195

E-mail address: `jkim@amath.washington.edu`

DEPARTMENT OF APPLIED MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98195

E-mail address: `rjl@amath.washington.edu`