# High Resolution Methods and Adaptive Refinement for Tsunami Propagation and Inundation.

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**Summary.** We describe the extension of high resolution finite volume methods and adaptive refinement for the shallow water equations in the context of tsunami modeling. Godunov-type methods have been used extensively for modeling the shallow water equations in many contexts, however, tsunami modeling presents some unique challenges that must be overcome. We describe some of the specific difficulties associated with tsunami modeling, and summarize some numerical approaches that we have used to overcome these challenges. For instance, we have developed a well-balanced Riemann solver that is appropriate in the deep ocean regime as well as robust in near-shore and dry regions. Additionally, we have extended adaptive refinement algorithms to this application. We briefly describe some of the modifications necessary for using these adaptive methods for tsunami modeling.

## 1 Introduction

The shallow water equations are a commonly accepted governing approximation for tsunami propagation in the deep ocean as well as in near-shore regions—including inundation. Because of difficulties in the deep ocean associated with preserving the delicate steady state, the physically relevant form of the shallow water equations—a hyperbolic system for depth and momentum—can be problematic for transoceanic or global-scale tsunami modeling. Because of this, tsunami modelers have often used alternative forms of the shallow water equations which are valid for smooth solutions. However, these systems, and the numerical methods used to solve them, are rarely suited for modeling tsunami inundation since this regime can include steep gradients and shocks. We describe some numerical methods that are promising for modeling the physically relevant conservative form of the shallow water equations in all regimes of tsunami flow as well as adaptive algorithms that allow bridging the diverse scales at which the different regimes of tsunami flow occur.

The amplitude of tsunamis in the deep ocean is on the order of centimeters, meaning that tsunamis are a tiny perturbation to the steady state—a motionless

body of water several kilometers deep. The steady state exists due to a nontrivial balance of momentum flux and a source term due to nonconstant bottom bathymetry. Modeling tsunami propagation accurately, therefore, demands resolving this perturbation against the background steady state. This requires well-balanced Riemann solvers (e.g. [7, 10, 13]) if Godunov methods are to be used for the conservative form of the equations. As tsunamis approach the shore, energy compression leads to more violent flow characteristics including shocks or turbulent bores in the inundation regime. Numerical schemes must be able to capture or track these shocks and additionally must be robust and accurate given the appearance of dry states in the near-shore regime. A novel approximate Riemann solver that we have developed to handle these different challenges is briefly described in Section 3.

The diverse regimes of tsunami flow also occur at diverse spatial scales. In the deep ocean, tsunami wavelength is several hundred kilometers. Modeling transoceanic propagation obviously requires a large computing domain, yet, as tsunamis approach the shore the wavelength is compressed to several hundred meters in the shallow coastal waters. In order to accurately model coastal inundation, a much finer computational grid is required than can feasibly be used at the global scale. Grid cells of several hundred meters or significantly less are necessary if near shore bathymetry features vary on a small scale. Additionally, because bathymetry focuses tsunami energy unpredictably, areas needing the most refinement cannot be easily determined a priori. Furthermore, tsunami waves, like those in other hyperbolic systems, are often highly localized at a given time but move throughout the computational domain. For these reasons we believe that adaptive refinement is warranted for efficient global-scale tsunami calculations. We will briefly describe the extension of adaptive refinement algorithms, developed more generally for hyperbolic problems by Berger, Colella, Oliger and LeVeque [4, 6, 5], to the specificities of tsunami modeling.

The methods and results described in this paper are implemented in TSUNAMICLAW an extension of the CLAWPACK [11] software. We will describe the one-dimensional equations and algorithms for simplicity.

#### 2 The Shallow Water Equations

The shallow water equations are a commonly used approximation for modeling tsunamis in all regimes—from the deep ocean to the inundation regime. We solve the physically relevant hyperbolic system

$$h_t + (hu)_x = 0, (1)$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = -ghb_x,$$
 (2)

where h(x,t) is the fluid depth, u(x,t) the horizontal fluid velocity and b(x) the variable bottom bathymetry. We will use  $\eta$  to denote the fluid surface elevation

$$\eta(x,t) = h(x,t) + b(x).$$
 (3)

The physically relevant steady state, against which tsunamis propagate in the deep ocean, comes from the nontrivial balance of the pressure flux and the source term due to bathymetry

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$$\left(\frac{1}{2}gh^2\right)_x \approx -ghb_x. \tag{4}$$

It is well known (e.g. [10, 13]) that fractional step methods fail at preserving such steady states precisely. Therefore, considerable research has gone into developing well-balanced methods (e.g. [2, 3, 8, 10]) which preserve important discrete steady-states, such as the "ocean at rest" steady state, where  $\eta_x \equiv 0$  and  $u \equiv 0$ . When using adaptive refinement, it is equally important to maintain these steady states upon interpolating from coarse grids to fine grids, and averaging fine grids onto coarse grids.

# 3 Approximate Riemann Solvers for Wave Propagation Algorithms

The numerical method we use for this problem is based on the wave propagation algorithms described in [12]. These algorithms belong to the class of high-resolution Godunov-type finite volume methods that make use of Riemann problems at grid cell interfaces in order to solve hyperbolic systems of the form

$$q_t + f(q)_x = 0, (5)$$

 $q \in \mathbb{R}^m$ ,  $f(q) \in \mathbb{R}^m$ . A crucial component of the wave propagation algorithm is the determination of updating waves by a decomposition of the Riemann initial data into some set of appropriately chosen vectors

$$Q_i^n - Q_{i-1}^n = \sum_{p=1}^{M_w} \alpha_{i-1/2}^p r_{i-1/2}^p, \tag{6}$$

where  $Q_i^n \in \mathbb{R}^m$  is the numerical solution in the *i*<sup>th</sup> grid cell at time  $t^n$ ,  $M_w$  is the number of vectors used to approximate the Riemann solution and  $r^p \in \mathbb{R}^m$ ,  $p = 1, \ldots, M_w$ , are vectors chosen for the decomposition. Once the "waves"  $\alpha_{i-1/2}^p r_{i-1/2}^p$  are determined, each wave propagates at a chosen speed  $s_{i-1/2}^p$ . Of course, one usually chooses  $M_w = m$ , in order to have a unique solution for the scalars  $\alpha_{i-1/2}^p$ ,  $p = 1, \ldots, M_w$ . Typically  $\{r_{i-1/2}^p, s_{i-1/2}^p\}$  is some approximation to the  $p^{\text{th}}$  eigenpair of some locally linearized Jacobian:  $\bar{A}(Q_{i-1}^n, Q_i^n) \approx f'(Q_{i-1}^n) \approx f'(Q_i^n)$ .

A consistent and alternative method to (6) for determining waves of an approximate Riemann solution is to decompose the flux into a set of propagating vectors

$$f(Q_i^n) - f(Q_{i-1}^n) = \sum_{p=1}^{M_w} \beta_{i-1/2}^p r_{i-1/2}^p,$$
(7)

where again  $\{r_{i-1/2}^{p}, s_{i-1/2}^{p}\}$  is some approximation to the the  $p^{\text{th}}$  eigenpair of some linearized Jacobian. In this case  $\beta_{i-1/2}^{p}$  has the same dimension as  $s_{i-1/2}^{p}\alpha_{i-1/2}^{p}$ . This method, described more fully in [3], has the advantageous property of producing conservative Riemann solutions regardless of the form of the vectors  $r_{i-1/2}^{p}$ . If the eigenpairs of a Roe averaged Jacobian are used to define  $r_{i-1/2}^{p}$  and  $s_{i-1/2}^{p}$  in (6) and (7), the two methods are the same.

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For nonhomogeneous hyperbolic systems

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$$q_t + f(q)_x = \psi(q, x), \tag{8}$$

it is consistent to include the effect of a source term directly into updating waves by performing the decomposition

$$f(Q_i^n) - f(Q_{i-1}^n) - \Psi_{i-1/2}\Delta x = \sum_{p=1}^{M_w} \beta_{i-1/2}^p r_{i-1/2}^p, \tag{9}$$

where  $\Psi_{i-1/2}$  is some consistent approximation to the source term  $\psi(q, x)$  at  $x_{i-1/2}$ , and  $\Delta x = x_i - x_{i-1}$ . See [3] for more details.

In [14], LeVeque and Pelanti explore the idea of performing a decomposition of the form

$$\begin{bmatrix} Q_i^n - Q_{i-1}^n \\ f(Q_i^n) - f(Q_{i-1}^n) \end{bmatrix} = \sum_{p=1}^{2m} \alpha_{i-1/2}^p \begin{bmatrix} w_{i-1/2}^p \\ \phi_{i-1/2}^p \end{bmatrix},$$
(10)

using vectors  $(w_{i-1/2}^p, \phi_{i-1/2}^p)^T \in \mathbb{R}^{2m}$ , for  $p = 1, \ldots, 2m$ . Several options for  $w_{i-1/2}^p$ and  $\phi_{i-1/2}^p$  are explored, and connections to relaxation solvers are explored. For instance, it is possible to choose  $(w_{i-1/2}^p, \phi_{i-1/2}^p)^T$  such that the methods (6), (7) and (10) are all the same.

For the shallow water equations (1), we have developed a solver based on a decomposition of the form

$$\begin{bmatrix} Q_i^n - Q_{i-1}^n \\ \varphi(Q_i^n) - \varphi(Q_{i-1}^n) \\ b_i - b_{i-1} \end{bmatrix} = \sum_{p=0}^3 \alpha_{i-1/2}^p \tilde{r}_{i-1/2}^p,$$
(11)

where  $\tilde{r}^p \in \mathbb{R}^4$ , for p = 0, ..., 3,  $q = (h, hu)^T$ ,  $\varphi(q) = (hu^2 + \frac{1}{2}gh^2)$  and  $b_i$  is the bathymetry in the *i*<sup>th</sup> grid cell. The vectors  $\tilde{r}^p_{i-1/2}$  and corresponding propagation speeds  $\tilde{s}^p_{i-1/2}$  are chosen to be local approximations to the eigenvectors and eigenvalues of  $\tilde{A}(\tilde{q}) \in \mathbb{R}^{4 \times 4}$ , where

$$\tilde{q}_t + \tilde{A}(\tilde{q})\tilde{q}_x = 0, \tag{12}$$

is an over-determined system equivalent to the shallow water equations for smooth solutions  $\tilde{q} = (h, hu, \varphi, b)^T$ . By choosing certain averages in the local approximations  $\tilde{r}_{i-1/2}^p$ , the method has certain desirable properties. For instance, since  $\tilde{s}_{i-1/2}^0 = 0$ , the solver preserves a large class of discrete steady state solutions as a stationary contact discontinuity at the cell interface  $x_{i-1/2}$ . Second, by defining the stationary eigenvector  $\tilde{r}_{i-1/2}^0$  appropriately, the solver preserves depth non-negativity of the approximate solution. Additionally, the solver can be shown to be equivalent to the Roe solver in the case of shock solutions over constant bathymetry. The Riemann solver is described in detail in [9].

## 4 Adaptive Mesh Refinement

To deal with the disparate spatial scales required to resolve a tsunami in the global propagation regime and the local inundation regime, we have extended adaptive

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mesh refinement routines (AMR) to this application. These algorithms (e.g. [4, 6, 5]) allow nesting of multiple rectangular Cartesian subgrids of several levels with integer refinement ratios within the computational domain. Because the subgrids evolve spatially and temporally, grids can essentially track moving features of the solution at various resolutions. For tsunami modeling, this allows transoceanic waves to be tracked on grids of suitable resolution without having to resolve unaffected regions of the ocean on fine grids. Additionally, since waves are compressed in the near shore region, even finer subgrids appear as waves approach the shore—allowing inundation modeling on meter-scale grids.

As mentioned, modeling tsunami propagation requires resolving a small deviation from the background steady state. Since AMR must interpolate data on coarse grids to generate finer level grids, and since fine grids must be averaged to update underlying coarse grid cells, special care must be taken to preserve steady states during this process. For instance, the standard approach of using a linear interpolant within each coarse cell, based on the conserved variables in the surrounding cells, does not in general preserve the common steady state  $\eta_x \equiv 0$ ,  $hu \equiv 0$  on the new fine grid. Similarly, averaging the conserved variables in fine grid cells contained within coarse cells does not preserve the steady state on the coarser grid. A simple one-dimensional example of interpolation from a coarse grid with a refinement ratio of 2 is shown in Figure 1. The steady-state is not maintained. Given practical grid resolutions for computations of the deep ocean, the spurious waves generated by such interpolation could be orders of magnitude larger than the actual tsunami being modeled.

A simple fix to the problem shown in Figure 1 is to interpolate the surface elevation  $\eta$  rather than the conserved variable h. The water depths on the fine grid are then determined from the values of  $\eta$  by  $h = \eta - b$ . It is easy to show that this maintains conservation of mass upon interpolating so long as all of the depths remain positive. Since maintaining conservation of momentum requires interpolating hu rather than u, it was necessary for us to develop a limiting strategy for determining fine grid values of hu to prevent unbounded velocities u on the fine grid. Additional strategies of interpolation had to be developed to prevent spurious "shore waves" upon interpolating grids near the shoreline. These issues are described in detail in [9].

An example TSUNAMICLAW simulation of the 2004 Indian Ocean TSUNAMI is shown in Figure 2. The tsunami was generated dynamically at the start of the computation by using a spatial temporal model of the fault motion provided by the Seismolab at Caltech [1].

## 5 Conclusions

Tsunamis exhibit diverse flow regimes requiring a numerical method that can simultaneously resolve near steady state solutions for transoceanic propagation as well as converge to shock solutions representing turbulent bores. Additionally the numerical method must be robust to the appearance of dry states in the inundation regime. We have developed an approximate Riemann solver that can handle these multiple features. Additionally the regimes of tsunamis occur at diverse spatial scales, requiring some form of grid refinement. We have modified adaptive refinement algorithms

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**Fig. 1.** Interpolating the water depth h from coarse grid cells to fine grid cells over nonlinear variable bathymetry. (a) A level l grid in one dimension for the common steady state problem of a motionless body of water with a flat surface  $\eta = 0$ . The numerical bathymetry is an average of the true bathymetry values in each grid cell. (b) Interpolation of h to a level (l+1) grid—refined by a factor of two—showing the two fine grid cells in the coarse cell  $C_i^l = [x_{i-1/2}, x_{i+1/2}]$ . The bathymetry in the two fine grid cells reflects the average of the true bathymetry in those cells. Interpolating the depth h using minmod slopes, destroys the steady state.



Fig. 2. TSUNAMICLAW simulation of the 2004 Indian Ocean Tsunami using adaptive refinement. The simulation had 4 levels of refinement with refinement ratios of 8, 8 and 64. This example features inundation modeling in Eastern India. Grid lines on the highest level of each figure are omitted for clarity. (a) The transoceanic tsunami waves are resolved on 2nd level grids. Undisturbed areas of the Indian Ocean are maintained on the very coarse 1st level grid. (b) As the waves approach Sri Lanka and Eastern India, 3rd level grids appear around the impacted shores. (c) The region around Madras, India is resolved on 4th level grids. This region is within the northernmost 3rd level grids shown in the upper right figure. (d) Inundation of the Madras, India area is resolved on fine-scale 4th level grids. The commercial harbor to the south suffers greater inundation than the fishing harbor visible to the north. A color version is available at http://www.amath.washington.edu/ rjl/pubs/hyp06tsunami.

for this application, so that transoceanic propagation and local inundation can be modeled in single global-scale computations.

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