Reduced Order Models For A Few Problems with Parameter Sweeps

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Outline

- Frequency Response Problem
  - Work in collaboration with C. Farhat and R. Tezaur (Stanford U.)
- Coherent Transport using Green function
  - Work in collaboration with M. P. Anantram and D. Ji (U. Washington - EE)
- Incompressible flow simulations with varying viscosity
  - Work in collaboration with Y. Wu (U. Washington)

Fluid-Structure Acoustic Vibrations Model

Discrete problems of the form

$$\begin{bmatrix} \mathbf{M}_s & \mathbf{0} \\ -\rho_f \mathbf{C} & \frac{1}{c_f^2}\mathbf{M}_f \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_s \\ \ddot{\mathbf{p}}_f \end{bmatrix} + \begin{bmatrix} \mathbf{D}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_f \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}}_s \\ \dot{\mathbf{p}}_f \end{bmatrix} + \begin{bmatrix} \mathbf{K}_s & \mathbf{C}^T \\ \mathbf{0} & \mathbf{K}_f \end{bmatrix} \begin{bmatrix} \mathbf{u}_s \\ \mathbf{p}_f \end{bmatrix} = \begin{bmatrix} \mathbf{f}_s \\ \mathbf{g}_f \end{bmatrix}$$

- Very large sparse matrices ($\geq 10^6$).
- $\mathbf{M}_s$, $\mathbf{K}_s$, $\mathbf{M}_f$, and $\mathbf{K}_f$ are usually real symmetric matrices.
- $\mathbf{D}_s$ and $\mathbf{D}_f$ are damping matrices.
- $\mathbf{C}$ is the coupling matrix between the structure and fluid.

These matrices typically depend on geometry, topology, and material parameters.

Frequency Response Problems

Time-harmonic excitation

$$\begin{bmatrix} \mathbf{f}_s \\ \mathbf{g}_f \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix} e^{i\omega t} \quad \Rightarrow \quad \begin{bmatrix} \mathbf{u}_s \\ \mathbf{p}_f \end{bmatrix} = \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} e^{i\omega t}$$

The model becomes a frequency-dependent linear system

$$-\omega^2 \begin{bmatrix} \mathsf{M}_s & 0 \\ -\rho_f \mathsf{C} & \frac{1}{c_f^2}\mathsf{M}_f \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} + i\omega \begin{bmatrix} \mathsf{D}_s & 0 \\ 0 & \mathsf{D}_f \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix}$$
$$+ \begin{bmatrix} \mathsf{K}_s & \mathsf{C}^T \\ 0 & \mathsf{K}_f \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}$$

Frequency Response Problems

Discrete problems of the form

$$(\mathsf{K} - \omega^2 \mathsf{M} + \mathsf{S}(\omega))\mathsf{x}(\omega) = \mathsf{f}(\omega)$$

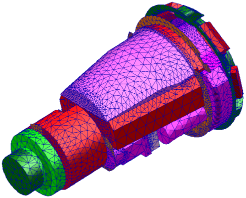
Frequency response problems have to be solved for many parameters.

- Very large sparse matrices ($\geq 10^6$)
- Problem with multiple frequencies (100+ ω) in an interval $[\omega_l, \omega_r]$
- Interested in the whole field $\mathsf{x}(\omega)$
- Linear systems are increasingly difficult to solve when ω grows

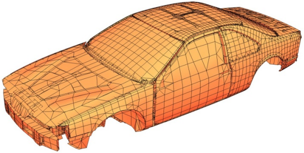
Applications of Frequency Response Problems

- Structural vibrations and interior noise acoustics

Elastodynamics

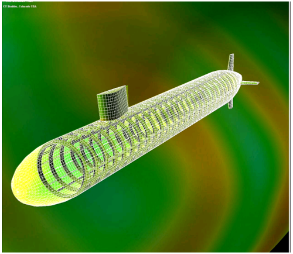


Interior Helmholtz

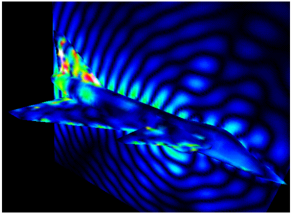


- Scattering

Exterior Helmholtz



Maxwell



Frequency Response Computation

- Straightforward Algorithm
  - Sample the interval of interest $[\omega_l, \omega_r]$: $\hat{\omega}_0, \cdots, \hat{\omega}_S$
  - Solve the linear system

$$(\mathbf{K} - \omega^2 \mathbf{M} + \mathbf{S}(\omega))\mathbf{x}(\omega) = \mathbf{f}(\omega)$$

- CPU intensive
- Reduced-order model can speed up the simulation time.

Reduced Order Model Techniques

- Many techniques have been proposed with a similar goal.
  - Reviews: Freund (2003), Bai-Dewilde-Freund (2005), Antoulas (2005), ...
- Interpolatory Reduced Order Models (Padé approximation or moment matching)
  - Freund (2003), Bai-Su (2005), Beattie-Gugercin (2005), Patera et al. (2006), Olsson-Ruhe (2006), Avery-Farhat-Reese (2007), Meerbergen (2008), Tuck Lee-Pinsky (2008), ...

Galerkin Projection

Consider

$$\mathbf{x}(\omega) = \left(\mathbf{K} - \omega^2\mathbf{M} + \mathbf{S}(\omega)\right)^{-1}\mathbf{f}(\omega)$$

A Galerkin approximation in the subspace $\mathbf{V}$ is such that

$$\mathbf{V}^H\left[\left(\mathbf{K} - \omega^2\mathbf{M} + \mathbf{S}(\omega)\right)\mathbf{V}\tilde{\mathbf{x}}(\omega) - \mathbf{f}(\omega)\right] = \mathbf{0}$$

$$\mathbf{V}^H\left(\mathbf{K} - \omega^2\mathbf{M} + \mathbf{S}(\omega)\right)\mathbf{V}\tilde{\mathbf{x}}(\omega) = \mathbf{V}^H\mathbf{f}(\omega)$$

$$\tilde{\mathbf{x}}(\omega) = \left(\mathbf{V}^H\mathbf{K}\mathbf{V} - \omega^2\mathbf{V}^H\mathbf{M}\mathbf{V} + \mathbf{V}^H\mathbf{S}(\omega)\mathbf{V}\right)^{-1}\mathbf{V}^H\mathbf{f}(\omega)$$

$$\mathbf{x}(\omega) \approx \mathbf{V}\tilde{\mathbf{x}}(\omega) = \mathbf{V}\left(\mathbf{V}^H\mathbf{K}\mathbf{V} - \omega^2\mathbf{V}^H\mathbf{M}\mathbf{V} + \mathbf{V}^H\mathbf{S}(\omega)\mathbf{V}\right)^{-1}\mathbf{V}^H\mathbf{f}(\omega)$$

Interpolatory Approximation

A one-point matching approximation of x at ω_0 is defined as a function $\mathsf{x}_{(\omega_0;J)}$ satisfying

$$\mathsf{x}_{(\omega_0;J)}(\omega_0) = \mathsf{x}(\omega_0) \quad \frac{d^j \mathsf{x}_{(\omega_0;J)}}{d\omega^j}(\omega_0) = \frac{d^j \mathsf{x}}{d\omega^j}(\omega_0), \quad \forall j < J$$

The Taylor expansion of $\mathsf{x}_{(\omega_0;J)}$ around ω_0 matches the first J terms of the Taylor expansion of x around the same point,

$$\mathsf{x}(\omega) = \mathsf{x}_{(\omega_0;J)}(\omega) + \mathscr{O}\left((\omega - \omega_0)^J\right)$$

Local convergence around the expansion point ω_0.

Interpolatory Approximation

The derivatives of $\mathbf{x}$ are solutions of a system of linear equations with the same left-hand side

$$\left(\mathbf{K} - \omega^2\mathbf{M} + \mathbf{S}(\omega)\right)\frac{d\mathbf{x}}{d\omega}(\omega) = \frac{d\mathbf{f}}{d\omega}(\omega) - \left(\frac{d\mathbf{S}}{d\omega}(\omega) - 2\omega\mathbf{M}\right)\mathbf{x}(\omega)$$

and

$$\begin{aligned}\left(\mathbf{K} - \omega^2\mathbf{M} + \mathbf{S}(\omega)\right)\frac{d^j\mathbf{x}}{d\omega^j}(\omega) &= \frac{d^j\mathbf{f}}{d\omega^j}(\omega) \\ &\quad - \sum_{k=1}^{j}\frac{j!}{k!(j-k)!}\frac{d^k}{d\omega^k}\left(\mathbf{K} - \omega^2\mathbf{M} + \mathbf{S}(\omega)\right)\frac{d^{j-k}\mathbf{x}}{d\omega^{j-k}}(\omega)\end{aligned}$$

One-Point Matching via Galerkin Projection

Consider the matching frequency ω_0. If

$$\mathrm{span}\left\{\mathsf{x}(\omega_0), \frac{d\mathsf{x}}{d\omega}(\omega_0), \ldots, \frac{d^{J-1}\mathsf{x}}{d\omega^{J-1}}(\omega_0)\right\} \subset \mathrm{span}\mathbf{V}$$

Then,

$$\mathsf{x}(\omega) = \mathbf{V}\left(\mathbf{V}^H\mathbf{K}\mathbf{V} - \omega^2\mathbf{V}^H\mathbf{M}\mathbf{V} + \mathbf{V}^H\mathbf{S}(\omega)\mathbf{V}\right)^{-1}\mathbf{V}^H\mathbf{f}(\omega) + \mathcal{O}\left((\omega - \omega_0)^J\right)$$

- Freund-Feldmann (1995), Beattie-Gugercin (2009), ..

Interpolatory Model Reduction in a Frequency Band

A Galerkin projection-based method for approximating

$$\mathbf{x}(\omega) = \left(\mathbf{K} - \omega^2\mathbf{M} + \mathbf{S}(\omega)\right)^{-1}\mathbf{f}(\omega)$$

in a frequency interval consists in

- Building $\mathbf{V}$ that satisfies the property

$$\text{span}\left\{\mathbf{x}(\omega_0), \frac{d\mathbf{x}}{d\omega}(\omega_0), \ldots, \frac{d^{J-1}\mathbf{x}}{d\omega^{J-1}}(\omega_0)\right\} \subset \text{span}\mathbf{V}$$

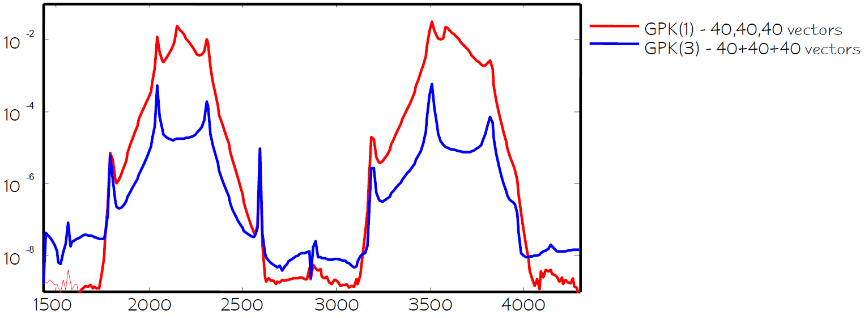
- Evaluating, in the frequency interval,

$$\tilde{\mathbf{x}}(\omega) = \left(\mathbf{V}^H\mathbf{K}\mathbf{V} - \omega^2\mathbf{V}^H\mathbf{M}\mathbf{V} + \mathbf{V}^H\mathbf{S}(\omega)\mathbf{V}\right)^{-1}\mathbf{V}^H\mathbf{f}(\omega)$$

- Approximating $\mathbf{x}(\omega)$ with $\mathbf{V}\tilde{\mathbf{x}}(\omega)$

Towards Multi-Point Matching

- A single point approximation converges first around the expansion point.
- To get a better approximation over an interval, multiple expansion points can be used at the same time.



- Example for frequency sweep analysis of a thick spherical steel shell submerged in water and excited by a point load on its inner surface.

Multi-Point Matching via Galerkin Projection

Consider the matching frequencies $\omega_0, \cdots, \omega_{P-1}$. If

$$\mathrm{span}\left\{\mathsf{x}(\omega_0), \frac{d\mathsf{x}}{d\omega}(\omega_0), \ldots, \frac{d^{J-1}\mathsf{x}}{d\omega^{J-1}}(\omega_0)\right\} + \ldots$$
$$\mathrm{span}\left\{\mathsf{x}(\omega_{P-1}), \frac{d\mathsf{x}}{d\omega}(\omega_{P-1}), \ldots, \frac{d^{J-1}\mathsf{x}}{d\omega^{J-1}}(\omega_{P-1})\right\} \subset \mathrm{span}\mathbf{V}$$

Then, formally,

$$\mathsf{x}(\omega) = \mathbf{V}\left(\mathbf{V}^H\mathbf{K}\mathbf{V} - \omega^2\mathbf{V}^H\mathbf{M}\mathbf{V} + \mathbf{V}^H\mathbf{S}(\omega)\mathbf{V}\right)^{-1}\mathbf{V}^H\mathbf{f}(\omega)$$
$$+ \mathscr{O}\left(\prod_{p=0}^{P-1}(\omega - \omega_p)^J\right)$$

- Gallivan-Grimme-van Dooren (1996), Beattie-Gugercin (2009), ...

Multi-Point Matching via Galerkin Projection

Consider

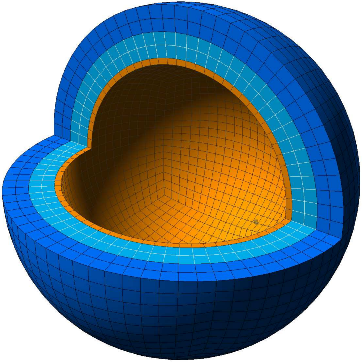
$$\mathsf{x}(\omega) = \left(\mathbf{K} - \omega^2 \mathbf{M} + \mathbf{S}(\omega)\right)^{-1} \mathbf{f}(\omega)$$

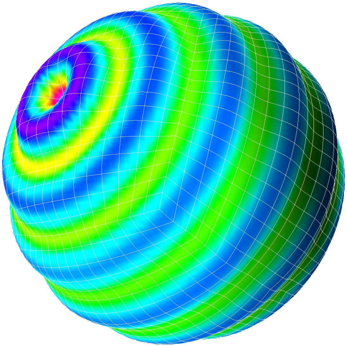
- Loop on frequencies: $\omega_0, \cdots, \omega_{P-1}$
  - Set $\mathbf{x}(\omega_p) = \left(\mathbf{K} - \omega_p^2 \mathbf{M} + \mathbf{S}(\omega_p)\right)^{-1} \mathbf{f}(\omega_p)$
  - Compute the derivatives of $\mathbf{x}$ at ω_p (systems with same matrix)
  - Augment $\mathbf{V}$ with these new derivatives
  - Form an orthonormal basis
- Compute the projected matrices: $\mathbf{V}^H\mathbf{K}\mathbf{V}$, $\mathbf{V}^H\mathbf{M}\mathbf{V}$, $\mathbf{V}^H\mathbf{S}(\omega)\mathbf{V}$
- Approximate the frequency response over $[\omega_l, \omega_r]$

$$\mathsf{x}(\omega) \approx \mathbf{V}\left(\mathbf{V}^H\mathbf{K}\mathbf{V} - \omega^2\mathbf{V}^H\mathbf{M}\mathbf{V} + \mathbf{V}^H\mathbf{S}(\omega)\mathbf{V}\right)^{-1}\mathbf{V}^H\mathbf{f}(\omega)$$

Fluid-Structure Acoustic Vibration Analysis

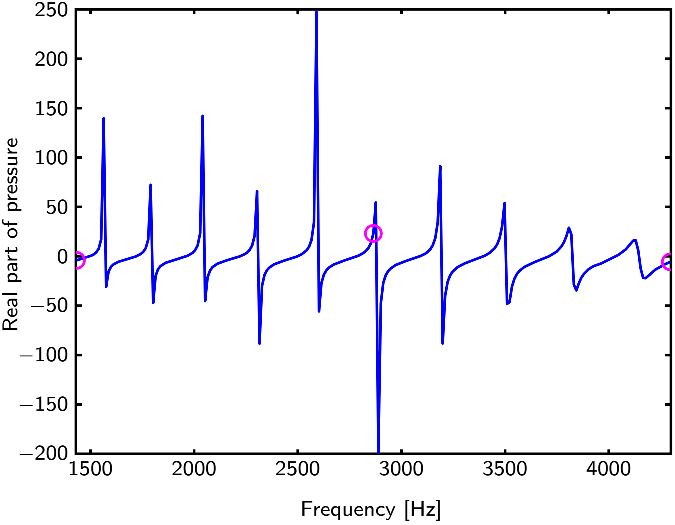
Frequency sweep analysis of a thick spherical steel shell submerged in water and excited by a point load on its inner surface.
FE model using isoparametric cubic elements incorporates a perfectly matched layer (PML) and roughly 400,000 dofs

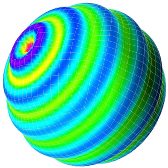




Fluid-Structure Acoustic Vibration Analysis

- Frequency sweep analysis of a submerged thick spherical shell

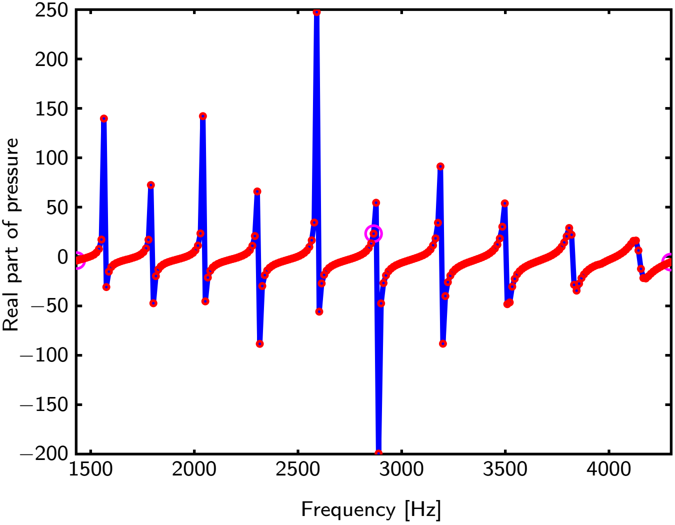


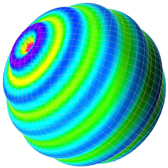


- Reference
- Coarse frequencies 1430Hz, 2860Hz, and 4290Hz

Fluid-Structure Acoustic Vibration Analysis

- Frequency sweep analysis of a submerged thick spherical shell

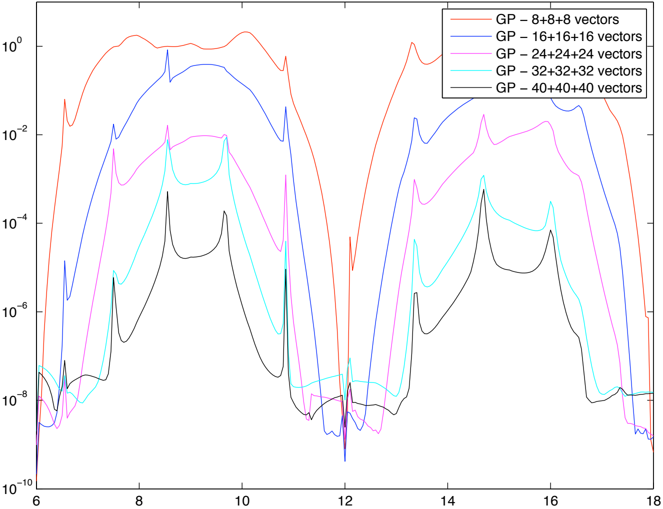




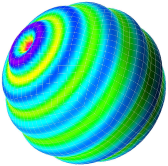
- Reference
- Coarse frequencies 1430Hz, 2860Hz, and 4290Hz
- 3-point GPK with 32+32+32 vectors
- Sampling every 12Hz (240 samples)

Fluid-Structure Acoustic Vibration Analysis

- Frequency sweep analysis of a submerged thick spherical shell



- 3-point GPK
- FETI-DPH with tolerance 10^{-8}



Solutions of Linear Systems

- Iterative domain-decomposition algorithm for solving

$$(\mathbf{K} - \omega^2 \mathbf{M})\,\mathbf{x} = \mathbf{b}$$

  - **K** and **M** are complex symmetric matrices
  - FETI-DPH
  - GMRES on Lagrange multipliers

- Matching of 72 $(= 24 + 24 + 24)$ values at 3 points on 16 nodes (with 8 cores)
  - "Straightforward" sweep with FETI-DPH and 240 samples $\approx 240 \times 20s = 4,800s$
  - Average of 41 iterations per right hand side (128 subdomains, coarse grid size 30,039)
  - CPU time for reduced order model and "reduced" sweep: 313s

Motivation for Adaptivity

- The quality of interpolatory reduced-order models depends on the selection of interpolation points and on the number of matched derivatives.
- This selection has mostly been heuristic, which remains the main disadvantage of interpolatory model reduction.

1. Selecting Matching Points

- Starting from P_0 sampling frequencies, build a first model matching J derivatives per point.
- Monitor the residual norms

$$\omega \mapsto \frac{\left\| \left(\mathbf{K} - \omega^2 \mathbf{M} + \mathbf{S}(\omega)\right) \mathbf{V}\tilde{\mathbf{x}}(\omega) - \mathbf{f}(\omega) \right\|_2}{\|\mathbf{f}(\omega)\|_2}$$

  on every interval $[\omega_p, \omega_{p+1}] \subset [\omega_l, \omega_r]$.
- Add a new sampling frequency if one norm value is greater than tolerance τ.
  - Check values at a few points in $[\omega_p, \omega_{p+1}]$.
- Create a new interpolatory reduced order model (if needed).

2. Selecting Matching Points and Number of Derivatives

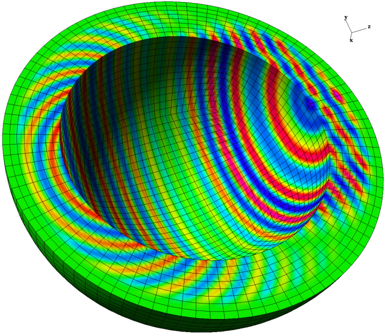
- Use the adaptive scheme for placing points described previously and monitoring the residual norms

$$\omega \mapsto \frac{\left\| \left(\mathsf{K} - \omega^2 \mathsf{M} + \mathsf{S}(\omega) \right) \mathsf{V} \tilde{\mathsf{x}}(\omega) - \mathsf{f}(\omega) \right\|_2}{\|\mathsf{f}(\omega)\|_2} \quad \text{on} \;\; [\omega_l, \omega_r]$$

- For every sampling frequency ω_p,
  - Compute $J_{\max}$ derivatives
  - Keep a number of derivatives between $J_{\min}$ and $J_{\max}$ such that
    - the residual norms decrease sufficiently with the new directions
    - the residual norms are just below tolerance τ

$(1 \leq J_{\min} \leq J_{\max})$

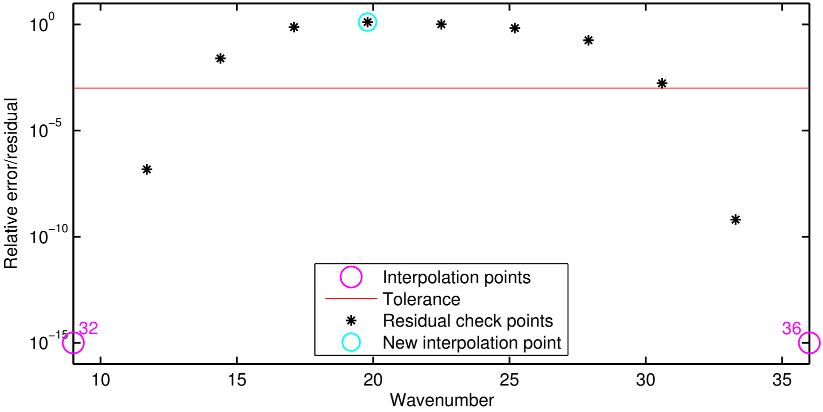
Example: Helmholtz Scattering Problem



$$(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{p}_f(\omega) = \mathbf{g}_f(\omega) \quad \omega \in [9, 36]$$

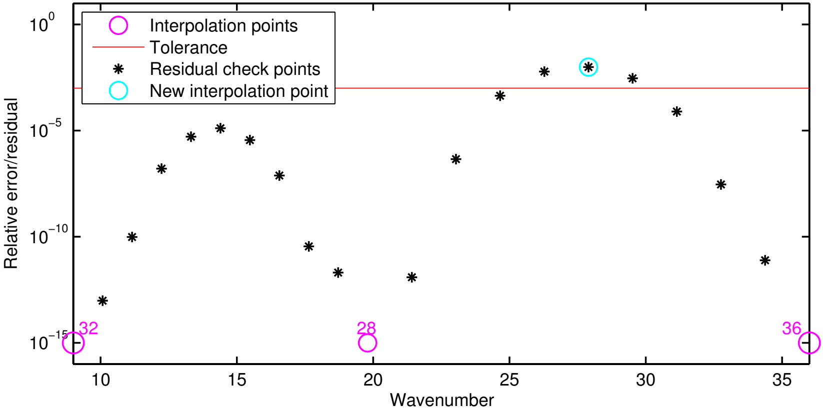
- $\mathbf{K}$ and $\mathbf{M}$ are complex symmetric matrices of size $777,650$ (cubic elements).
- A *frozen* perfectly matched layer is used.

Automatic Adaptivity



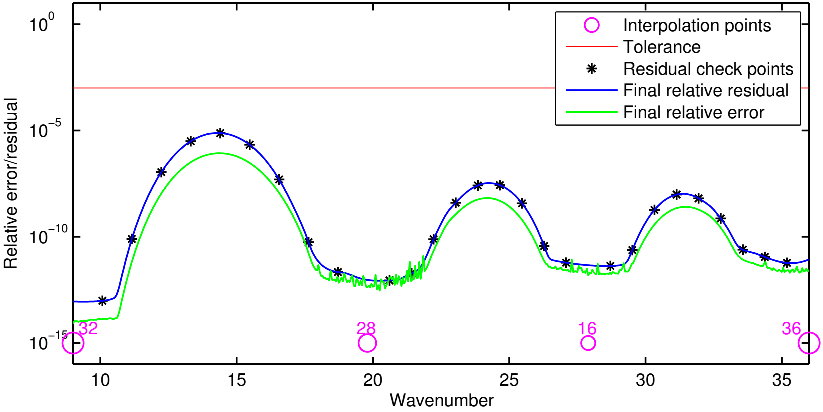
- $J_{min} = 16$ and $J_{max} = 64$.

Automatic Adaptivity



- $J_{min} = 16$ and $J_{max} = 64$.

Automatic Adaptivity



- $J_{min} = 16$ and $J_{max} = 64$.

Cost and Accuracy

Table I. Acoustic scattering problem: performance of the adaptive DGP method in terms of CPU cost and accuracy ($\tau = 10^{-3}$, 33 CPU cores).

$J_{min}, J_{max}, \Delta J$		8,8,0	16,16,0	32,32,0	48,48,0	64,64,0	16,64,4
# of interpolation points		7	5	4	4	4	4
Size of approximation subspace		56	80	128	192	256	112
Average relative residual		1.2e-7	7.2e-8	3.3e-7	2.1e-8	9.1e-8	5.2e-7
Maximum relative residual		1.9e-6	9.7e-7	4.9e-6	1.1e-7	1.4e-6	7.6e-6
Maximum relative error		5.9e-7	4.7e-7	2.0e-7	2.6e-7	1.6e-7	8.5e-7
CPU time (s)	Subspace construction	712.7	534.9	471.0	509.8	573.6	448.1
	Projection setup	23.7	27.8	49.5	95.3	154.6	49.9
	Post-processing	38.0	48.6	68.6	97.0	125.4	62.1
	Total	774.3	611.3	589.1	774.4	853.6	560.1

- Brute force time with 541 samples: 49,339 s

Non-Equilibrium Green's Functions (NEGF)

The retarded Green's function $\mathbf{G}^r$ is defined

$$\left(E\mathbf{I} - \mathbf{H} - \Sigma_{lead}\left(E\right)\right)\mathbf{G}^r\left(E\right) = \mathbf{A}\mathbf{G}^r \quad = \quad \mathbf{I}$$

The electron density at point q and energy E is

$$n_q(E) = 2\int \frac{dE}{2\pi} G^r_{qL}\Sigma^<_{LL}\left(G^r_{qL}\right)^H + G^r_{qR}\Sigma^<_{RR}\left(G^r_{qR}\right)^H$$

and the current from lead L to R is

$$J_{L\to R} = \frac{2e}{h}\int dE T_{LR}\left(E\right)\left(f_L\left(E\right) - f_R\left(E\right)\right)$$

$$T_{LR}\left(E\right) = \mathrm{tr}\left[\Gamma_{LL}\left(E\right)G^r_{LR}\left(E\right)\Gamma_{RR}\left(E\right)\left(G^r_{LR}\left(E\right)\right)^H\right]$$

Non-Equilibrium Green's Functions (NEGF)

The problem is to solve

$$(E\mathbf{I} - \mathbf{H} - \Sigma_{lead}(E))\,\mathbf{G}_L^r(E) = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

and

$$(E\mathbf{I} - \mathbf{H} - \Sigma_{lead}(E))\,\mathbf{G}_R^r(E) = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \end{bmatrix}$$

over a wide range of energy values.

Strategy

- Build an interpolatory reduced-order model to approximate columns of $\mathbf{G}^r$ over $[E_{\min}, E_{\max}]$
- Select the matching energies adaptively.
- Select the columns to interpolate adaptively.

Experiments

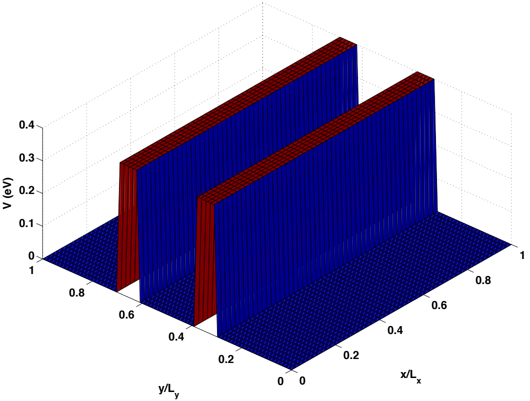


Figure: Potential for the rectangular nanodevice

Experiments

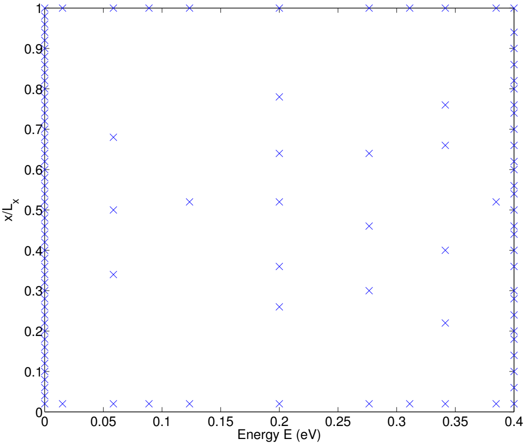


Figure: Sampling Values

Experiments

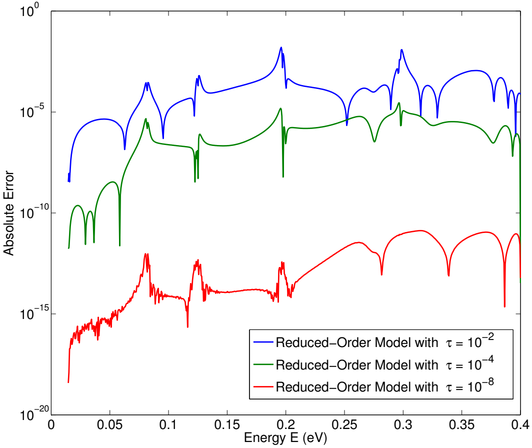


Figure: Absolute Error between FOM and ROM

Experiments

N_x	Size of FOM	Time Brute Force	Size of ROM	Time ROM
26	$n = 1014$	$23s$	$s = 86$	$7s$
50	$n = 3750$	$182s$	$s = 113$	$39s$
100	$n = 15000$	$1828s$	$s = 156$	$252s$

Table: Evolution of timings as the number of grid points N_x is increased (with evaluation of transmission at 768 energy points).

- Tolerance $\tau = 10^{-2}$
- $N_y = 3N_x/2$

Problem Formulation

Consider the incompressible Navier-Stokes equation in a bounded domain $\Omega \in \mathbb{R}^2$

$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \Delta \mathbf{u} &= \mathbf{f} \quad \text{in } \Omega \times (0, T] \\
\nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega \times (0, T] \\
\mathbf{u} &= \mathbf{b} \quad \text{on } \Gamma_D \times (0, T] \\
-p\mathbf{n} + \nu \nabla u \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma_N \times (0, T] \\
\mathbf{u}(0) &= \mathbf{u}_0
\end{aligned}
$$

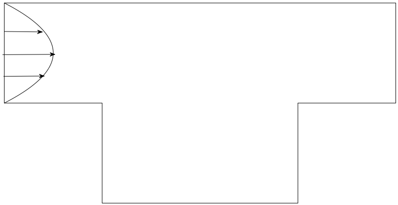
where $\mathbf{u}(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$ denote the velocity and pressure fileds. Ω is a bounded domain in $\mathbb{R}^2$ whose boundary is denoted by $\partial\Omega = \Gamma_D \cup \Gamma_N$. ν denote the kinematic viscosity of the fluid, $\mathbf{u}_0$ a given initial velocity, and $\mathbf{b}$ a specified boundary velocity.

Adaptive Selection

1. Build the snapshot database for two viscosity values $S = \bigcup_{j=1}^{n_x}\{\mathbf{x}_j(\nu_1)\}\bigcup_{j=1}^{n_x}\{\mathbf{x}_j(\nu_P)\}$.
2. Set the number of clusters $N_V = 2$.
3. Construct the local ROBs Φ_i by partition S into N_v overlapping clusters
4. Evaluate the residuals at all the check points using ROM model
5. Find the viscosity value giving the largest residuals.
6. If the residuals are larger than a tolerance,
   - Set $S \leftarrow S \cup \{\mathbf{x}_1(\nu_{j_{max}}), \mathbf{x}_2(\nu_{j_{max}}), \ldots, \mathbf{x}_{n_x}(\nu_{j_{max}})\}$
   - $N_V \leftarrow N_V + 1$
   - Go to Step 3

Numerical experiments

- Incompressible Navier-Stokes flow on a T-shaped domain



- Inlet velocity $\mathbf{b} = \begin{bmatrix} 100(1+4sin(20\pi t))(1-y)(y-0.5) \\ 0 \end{bmatrix}$ for $0 \leq t \leq 0.05$
- Full order model: 11,203 degrees of freedom

Solution at training configuration $\nu = 1$

- $S = \{\mathbf{x}^j(\nu = 1)\}_{j=1}^{n_x}$ and $n_x = 500$

Approach	Error	Speedup
ROM	2.5e-04	1.71
ROM + 2 Clusters	5.8e-05	1.61
ROM + 3 Clusters	2.6e-05	1.51
ROM + NL Reduction	3.0e-04	104
ROM + 2 Clusters + NL Reduction	6.7e-05	106
ROM + 2 Clusters + NL Reduction	3.3e-05	100

Table: Performance of ROM of dimension 15.

Predictive ROM simulation

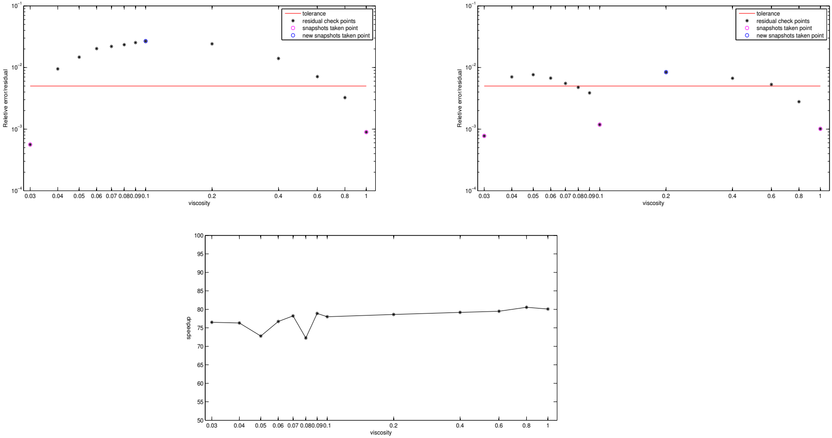


Figure: Parameter sweep for the T-shaped problem: POD size = 30

Summary

- We studied interpolatory reduced-order model.
- We designed adaptive schemes for selecting the matching points.