Outline

- Scalar nonlinear conservation laws
- Traffic flow
- Shocks and rarefaction waves
- Burgers' equation
- Rankine-Hugoniot conditions
- Importance of conservation form
- Weak solutions

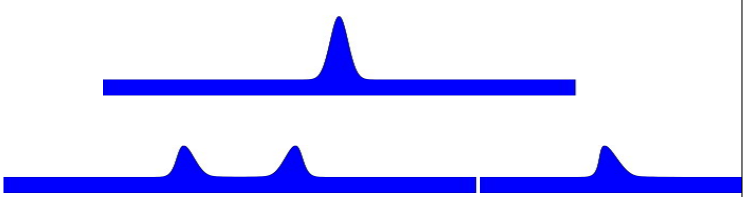
Reading: Chapter 11, 12

Notes:

Shock formation

For nonlinear problems wave speed generally depends on q.

Waves can steepen up and form shocks
$\Longrightarrow$ even smooth data can lead to discontinuous solutions.



Note:

- System of two equations gives rise to 2 waves.
- Each wave behaves like solution of nonlinear scalar equation.

Notes:

Shocks in traffic flow

Notes:

Car following model

$X_j(t)$ = location of jth car at time t on one-lane road.

$$\frac{dX_j(t)}{dt} = V_j(t).$$

Velocity $V_j(t)$ of jth car varies with j and t.

Simple model: Driver adjusts speed (instantly) depending on distance to car ahead.

$$V_j(t) = v\big(X_{j+1}(t) - X_j(t)\big)$$

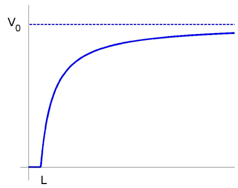
for some function $v(s)$ that defines speed as a function of separation s.

Simulations: **http://www.traffic-simulation.de/**

Notes:

R.J. LeVeque, University of Washington IPDE 2011, June 30, 2011 [FVMHP Chap. 11]

Function $v(s)$ (speed as function of separation)



$$v(s) = \begin{cases} u_{\max}\left(1 - \frac{L}{s}\right) & \text{if } s \geq L, \\ 0 & \text{if } s \leq L. \end{cases}$$

where:
L = car length
$u_{\max}$ = maximum velocity

Local density: $0 < L/s \leq 1$ $\quad(s = L \implies$ bumper-to-bumper)

R.J. LeVeque, University of Washington IPDE 2011, June 30, 2011 [FVMHP Chap. 11]

Notes:

R.J. LeVeque, University of Washington IPDE 2011, June 30, 2011 [FVMHP Chap. 11]

Continuum model

Switch to density function:

Let $q(x,t)$ = density of cars, normalized so:

Units for x: carlengths, so $x = 10$ is 10 carlengths from $x = 0$.

Units for q: cars per carlength, so $0 \leq q \leq 1$.

Total number of cars in interval $x_1 \leq x \leq x_2$ at time t is

$$\int_{x_1}^{x_2} q(x,t)\,dx$$

R.J. LeVeque, University of Washington IPDE 2011, June 30, 2011 [FVMHP Chap. 11]

Notes:

R.J. LeVeque, University of Washington IPDE 2011, June 30, 2011 [FVMHP Chap. 11]

Flux function for traffic

$q(x,t) =$ density, $u(x,t) =$ velocity $= U(q(x,t))$.

flux: $f(q) = uq$ Conservation law: $q_t + f(q)_x = 0$.

Constant velocity $u_{\max}$ independent of density:

$$f(q) = u_{\max} q \implies q_t + u_{\max} q_x = 0 \quad \text{(advection)}$$

Velocity varying with density:

$$V(s) = u_{\max}(1 - L/s) \implies U(q) = u_{\max}(1-q),$$
$$f(q) = u_{\max} q(1-q) \quad \text{(quadratic nonlinearity)}$$

Notes:

Characteristics for a scalar problem

$q_t + f(q)_x = 0 \implies q_t + f'(q) q_x = 0$ (if solution is smooth).

Characteristic curves satisfy $X'(t) = f'(q(X(t),t)), \quad X(0) = x_0$.

How does solution vary along this curve?

$$\begin{aligned}\frac{d}{dt} q(X(t),t) &= q_x(X(t),t)X'(t) + q_t(X(t),t) \\ &= q_x(X(t),t) f(q(X(t),t)) + q_t(X(t),t) \\ &= 0\end{aligned}$$

So solution is constant on characteristic
as long as solution stays smooth.

$q(X(t),t) =$ constant $\implies X'(t)$ is constant on characteristic,
so characteristics are straight lines!

Notes:

Nonlinear Burgers' equation

Conservation form: $u_t + \left(\frac{1}{2}u^2\right)_x = 0, \qquad f(u) = \frac{1}{2}u^2$.

Quasi-linear form: $u_t + u u_x = 0$.

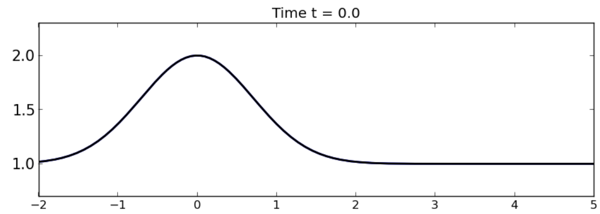
This looks like an advection equation with u advected with speed u.

True solution: u is constant along characteristic with speed $f'(u) = u$ until the wave "breaks" (shock forms).

Notes:

Burgers' equation

The solution is constant on characteristics so each value advects at constant speed equal to the value...



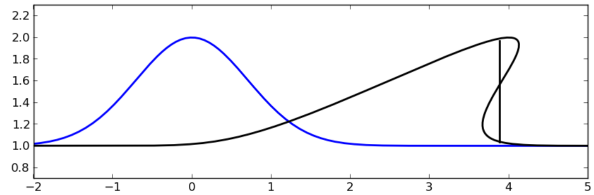
Notes:

Burgers' equation

Equal-area rule:

The area "under" the curve is conserved with time,

We must insert a shock so the two areas cut off are equal.



Notes:

Vanishing Viscosity solution

Viscous Burgers' equation: $u_t + \left(\frac{1}{2}u^2\right)_x = \epsilon u_{xx}$.

This parabolic equation has a smooth C^∞ solution for all $t > 0$ for any initial data.

Limiting solution as $\epsilon \to 0$ gives the shock-wave solution.

Why try to solve hyperbolic equation?

- Solving parabolic equation requires implicit method,
- Often correct value of physical "viscosity" is very small, shock profile that cannot be resolved on the desired grid
  $\Longrightarrow$ smoothness of exact solution doesn't help!

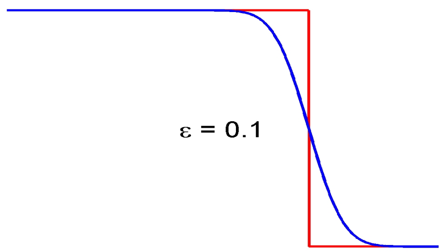
Notes:

Discontinuous solutions

Vanishing Viscosity solution: The Riemann solution $q(x,t)$ is the limit as $\epsilon \to 0$ of the solution $q^\epsilon(x,t)$ of the parabolic advection-diffusion equation

$$q_t + uq_x = \epsilon q_{xx}.$$

For any $\epsilon > 0$ this has a classical smooth solution:



Notes:

Weak solutions to $q_t + f(q)_x = 0$

$q(x,t)$ is a weak solution if it satisfies the integral form of the conservation law over all rectangles in space-time,

$$\int_{x_1}^{x_2} q(x,t_2)\,dx - \int_{x_1}^{x_2} q(x,t_1)\,dx$$
$$= \int_{t_1}^{t_2} f(q(x_1,t))\,dt - \int_{t_1}^{t_2} f(q(x_2,t))\,dt$$

Obtained by integrating

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = f(q(x_1,t)) - f(q(x_2,t))$$

from t_n to t_{n+1}.

Notes:

Weak solutions to $q_t + f(q)_x = 0$

Alternatively, multiply PDE by smooth test function $\phi(x,t)$, with compact support ($\phi(x,t) \equiv 0$ for $|x|$ and t sufficiently large), and then integrate over rectangle,

$$\int_0^\infty \int_{-\infty}^\infty (q_t + f(q)_x)\phi(x,t)\,dx\,dt$$

Then we can integrate by parts to get

$$\int_0^\infty \int_{-\infty}^\infty (q\phi_t + f(q)\phi_x)\,dx\,dt = -\int_0^\infty q(x,0)\phi(x,0)\,dx.$$

$q(x,t)$ is a weak solution if this holds for all such ϕ.

Notes:

Weak solutions to $q_t + f(q)_x = 0$

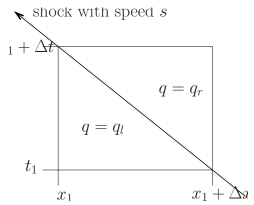
A function $q(x,t)$ that is piecewise smooth with jump discontinuities is a weak solution only if:

- The PDE is satisfied where q is smooth,
- The jump discontinuities all satisfy the Rankine-Hugoniot conditions.

Note: The weak solution may not be unique!

Notes:

Shock speed with states q_l and q_r at instant t_1



Then

$$\int_{x_1}^{x_1+\Delta x} q(x, t_1 + \Delta t)\, dx - \int_{x_1}^{x_1+\Delta x} q(x, t_1)\, dx$$
$$= \int_{t_1}^{t_1+\Delta t} f(q(x_1, t))\, dt - \int_{t_1}^{t_1+\Delta t} f(q(x_1 + \Delta x, t))\, dt.$$

Since q is essentially constant along each edge, this becomes

$$\Delta x\, q_l - \Delta x\, q_r = \Delta t f(q_l) - \Delta t f(q_r) + \mathcal{O}(\Delta t^2),$$

Taking the limit as $\Delta t \to 0$ gives

$$s(q_r - q_l) = f(q_r) - f(q_l).$$

Notes:

Rankine-Hugoniot jump condition

$$s(q_r - q_l) = f(q_r) - f(q_l).$$

This must hold for any discontinuity propagating with speed s, even for systems of conservation laws.

For scalar problem, any jump allowed with speed:

$$s = \frac{f(q_r) - f(q_l)}{q_r - q_l}.$$

For systems, $q_r - q_l$ and $f(q_r) - f(q_l)$ are vectors, s scalar,

R-H condition: $f(q_r) - f(q_l)$ must be scalar multiple of $q_r - q_l$.

For linear system, $f(q) = Aq$, this says

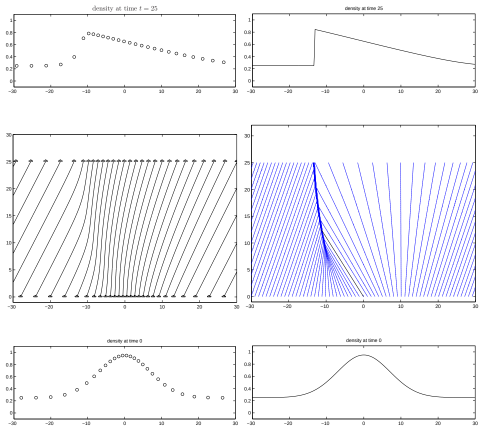
$$A(q_r - q_l) = s(q_r - q_l),$$

Jump must be an eigenvector, speed s the eigenvalue.

Notes:

Figure 11.1 — Shock formation in traffic

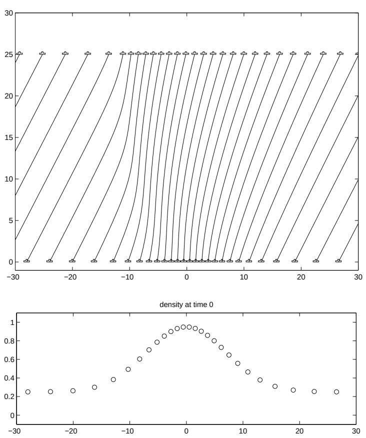
Discrete cars: Continuum model: $f'(q) = u_{\max}(1 - 2q)$



Notes:

Figure 11.1 — Shock formation

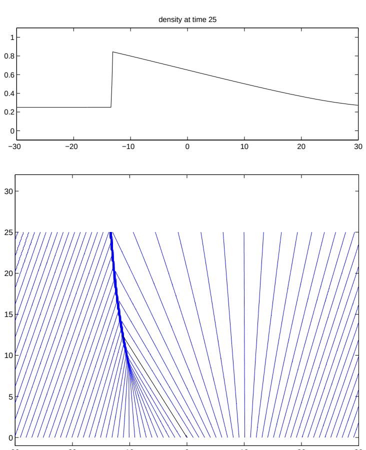
(a) particle paths (car trajectories) $u(x,t) = u_{\max}(1 - q(x,t))$



Notes:

Figure 11.1 — Shock formation

(b) characteristics: $f'(q) = u_{\text{max}}(1 - 2q)$



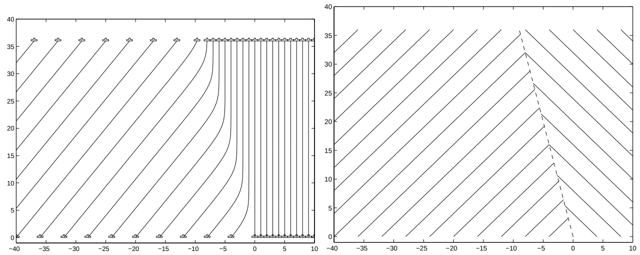
Notes:

Figure 11.2 — Traffic jam shock wave

Cars approaching red light $(q_\ell < 1, \;\; q_r = 1)$

Shock speed:

$$s = \frac{f(q_r) - f(q_\ell)}{q_r - q_\ell} = \frac{-2u_{\max} q_\ell}{1 - q_\ell} < 0.$$



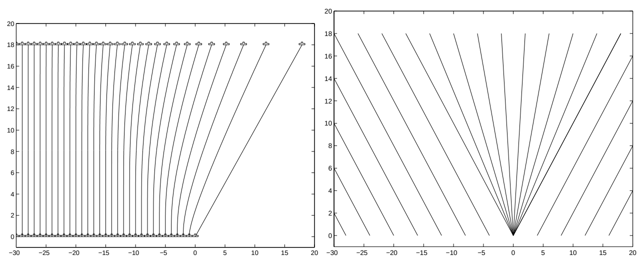
Notes:

Figure 11.3 — Rarefaction wave

Cars accelerating at green light $(q_\ell = 1, \;\; q_r = 0)$

Characteristic speed $f'(q) = u_{\max}(1 - 2q)$

varies from $f'(q_\ell) = -u_{\max}$ to $f'(q_r) = u_{\max}$.



Notes:

Nonlinear scalar conservation laws

Burgers' equation: $u_t + \left(\frac{1}{2}u^2\right)_x = 0.$

Quasilinear form: $u_t + uu_x = 0.$

These are equivalent for smooth solutions, not for shocks!

Upwind methods for $u > 0$:

Conservative: $U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}((U_i^n)^2 - (U_{i-1}^n)^2)\right)$

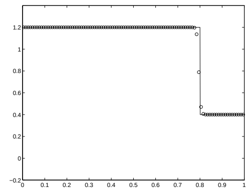
Quasilinear: $U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x} U_i^n (U_i^n - U_{i-1}^n).$

Ok for smooth solutions, not for shocks!

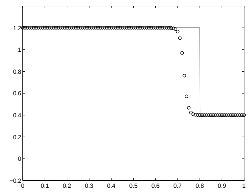
Notes:

Importance of conservation form

Solution to Burgers' equation using conservative upwind:



Solution to Burgers' equation using quasilinear upwind:



Notes:

Weak solutions depend on the conservation law

The conservation laws

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0$$

and

$$\left(u^2\right)_t + \left(\frac{2}{3}u^3\right)_x = 0$$

both have the same quasilinear form

$$u_t + uu_x = 0$$

but have different weak solutions,

different shock speeds!

Notes:

Conservation form

The method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

is in conservation form.

The total mass is conserved up to fluxes at the boundaries:

$$\Delta x \sum_i Q_i^{n+1} = \Delta x \sum_i Q_i^n - \frac{\Delta t}{\Delta x}(F_{+\infty} - F_{-\infty}).$$

Note: an isolated shock must travel at the right speed!

Notes:

Lax-Wendroff Theorem

Suppose the method is conservative and consistent with $q_t + f(q)_x = 0$,

$$F_{i-1/2} = \mathcal{F}(Q_{i-1}, Q_i) \quad \text{with } \mathcal{F}(\bar{q}, \bar{q}) = f(\bar{q})$$

and Lipschitz continuity of $\mathcal{F}$.

If a sequence of discrete approximations converge to a function $q(x,t)$ as the grid is refined, then this function is a weak solution of the conservation law.

Note:

Does not guarantee a sequence converges

Two sequences might converge to different weak solutions.

Also need stability and entropy condition.

Notes:

Non-uniqueness of weak solutions

For scalar problem, any jump allowed with speed:

$$s = \frac{f(q_r) - f(q_l)}{q_r - q_l}.$$

So even if $f'(q_r) < f'(q_l)$ the integral form of cons. law is satisfied by a discontinuity propogating at the R-H speed.

In this case there is also a rarefaction wave solution.

In fact, infinitely many weak solutions.

Which one is physically correct?

Notes:

Vanishing viscosity solution

We want $q(x,t)$ to be the limit as $\epsilon \to 0$ of solution to

$$q_t + f(q)_x = \epsilon q_{xx}.$$

This selects a unique weak solution:

- Shock if $f'(q_l) > f'(q_r)$,
- Rarefaction if $f'(q_l) < f'(q_r)$.

Lax Entropy Condition:

A discontinuity propagating with speed s in the solution of a convex scalar conservation law is admissible only if $f'(q_l) > s > f'(q_r)$.

Notes: