Shock formation

For nonlinear problems wave speed generally depends on \( q \).

Waves can steepen up and form shocks
\[ \implies \text{even smooth data can lead to discontinuous solutions.} \]

Note:
- System of two equations gives rise to 2 waves.
- Each wave behaves like solution of nonlinear scalar equation.
Car following model

\[ X_j(t) = \text{location of } j\text{th car at time } t \text{ on one-lane road.} \]

\[ \frac{dX_j(t)}{dt} = V_j(t). \]

Velocity \( V_j(t) \) of \( j\)th car varies with \( j \) and \( t \).

**Simple model:** Driver adjusts speed (instantly) depending on distance to car ahead.

\[ V_j(t) = v(X_{j+1}(t) - X_j(t)) \]

for some function \( v(s) \) that defines speed as a function of separation \( s \).

Simulations: http://www.traffic-simulation.de/

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Function \( v(s) \) (speed as function of separation)

\[ v(s) = \begin{cases} 
  u_{\text{max}} \left( 1 - \frac{s}{L} \right) & \text{if } s \geq L, \\
  0 & \text{if } s \leq L.
\end{cases} \]

where:

- \( L \) = car length
- \( u_{\text{max}} \) = maximum velocity

Local density: \( 0 < L/s \leq 1 \) \( (s = L \implies \text{bumper-to-bumper}) \)

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Continuum model

Switch to density function:

Let \( q(x, t) = \) density of cars, normalized so:

Units for \( x \): carlengths, so \( x = 10 \) is 10 carlengths from \( x = 0 \).

Units for \( q \): cars per carlength, so \( 0 \leq q \leq 1 \).

Total number of cars in interval \( x_1 \leq x \leq x_2 \) at time \( t \) is

\[ \int_{x_1}^{x_2} q(x, t) \, dx \]
Flux function for traffic

\[ q(x,t) = \text{density}, \quad u(x,t) = \text{velocity} = U(q(x,t)). \]

flux: \( f(q) = uq \)

Conservation law: \( q_t + f(q)_x = 0 \).

Constant velocity \( u_{\text{max}} \) independent of density:

\[ f(q) = u_{\text{max}}q \implies q_t + u_{\text{max}}q_x = 0 \quad \text{(advection)} \]

Velocity varying with density:

\[ V(s) = u_{\text{max}}(1 - L/s) \implies U(q) = u_{\text{max}}(1 - q), \]

\[ f(q) = u_{\text{max}}q(1 - q) \quad \text{(quadratic nonlinearity)} \]

Characteristics for a scalar problem

\[ q_t + f(q)_x = 0 \implies q_t + f'(q)q_x = 0 \quad \text{(if solution is smooth).} \]

Characteristic curves satisfy \( X'(t) = f(q(X(t), t)), \quad X(0) = x_0 \).

How does solution vary along this curve?

\[
\frac{dq(X(t), t)}{dt} = q_t(X(t), t)X'(t) + q_t(X(t), t) \\
= q_t(X(t), t)f(q(X(t), t)) + q_t(X(t), t) \\
= 0
\]

So solution is constant on characteristic as long as solution stays smooth.

\( q(X(t), t) = \text{constant} \implies X'(t) \) is constant on characteristic, so characteristics are straight lines!

Nonlinear Burgers’ equation

Conservation form: \( u_t + \left( \frac{1}{2} u^2 \right)_x = 0, \quad f(u) = \frac{1}{2} u^2. \)

Quasi-linear form: \( u_t + uu_x = 0. \)

This looks like an advection equation with \( u \) advected with speed \( u \).

True solution: \( u \) is constant along characteristic with speed \( f'(u) = u \) until the wave “breaks” (shock forms).
Burgers’ equation

The solution is constant on characteristics so each value advects at constant speed equal to the value...

![Graph showing the solution at time t = 0.0]

Notes:

Burgers’ equation

Equal-area rule:
The area “under” the curve is conserved with time,
We must insert a shock so the two areas cut off are equal.

![Graph showing equal-area rule]

Notes:

Vanishing Viscosity solution

Viscous Burgers’ equation: \( u_t + \left( \frac{1}{2} u^2 \right)_x = \epsilon u_{xx} \).

This parabolic equation has a smooth \( C^\infty \) solution for all \( t > 0 \) for any initial data.

Limiting solution as \( \epsilon \to 0 \) gives the shock-wave solution.

Why try to solve hyperbolic equation?

- Solving parabolic equation requires implicit method,
- Often correct value of physical “viscosity” is very small, shock profile that cannot be resolved on the desired grid \( \implies \) smoothness of exact solution doesn’t help!
Discontinuous solutions

**Vanishing Viscosity solution**: The Riemann solution \( q(x,t) \) is the limit as \( \epsilon \to 0 \) of the solution \( q^\epsilon(x,t) \) of the parabolic advection-diffusion equation

\[
q_t + uq_x = \epsilon q_{xx}.
\]

For any \( \epsilon > 0 \) this has a classical smooth solution:

- **Weak solutions to** \( q_t + f(q)_x = 0 \)

\( q(x,t) \) is a weak solution if it satisfies the integral form of the conservation law over all rectangles in space-time,

\[
\int_{x_1}^{x_2} q(x,t_2) \, dx - \int_{x_1}^{x_2} q(x,t_1) \, dx = \int_{t_1}^{t_2} f(q(x_1,t)) \, dt - \int_{t_1}^{t_2} f(q(x_2,t)) \, dt
\]

Obtained by integrating

\[
\frac{d}{dt} \int_{x_1}^{x_2} q(x,t) \, dx = f(q(x_1,t)) - f(q(x_2,t))
\]

from \( t_n \) to \( t_{n+1} \).

Alternatively, multiply PDE by smooth test function \( \phi(x,t) \), with compact support \( (\phi(x,t) \equiv 0 \) for \( |x| \) and \( t \) sufficiently large), and then integrate over rectangle,

\[
\int_0^\infty \int_{-\infty}^{\infty} (q_t + f(q)_x) \phi(x,t) \, dx \, dt
\]

Then we can integrate by parts to get

\[
\int_0^\infty \int_{-\infty}^{\infty} (q \phi_t + f(q) \phi_x) \, dx \, dt = -\int_0^\infty q(x,0) \phi(x,0) \, dx.
\]

\( q(x,t) \) is a weak solution if this holds for all such \( \phi \).
Weak solutions to \( q_t + f(q)_x = 0 \)

A function \( q(x, t) \) that is piecewise smooth with jump discontinuities is a weak solution only if:

- The PDE is satisfied where \( q \) is smooth,
- The jump discontinuities all satisfy the Rankine-Hugoniot conditions.

Note: The weak solution may not be unique!

Shock speed with states \( q_l \) and \( q_r \) at instant \( t_1 \)

Then

\[
\int_{x_1}^{x_1 + \Delta x} q(x, t_1 + \Delta t) \, dx - \int_{x_1}^{x_1 + \Delta x} q(x, t_1) \, dx = \int_{t_1}^{t_1 + \Delta t} f(q(x_1, t)) \, dt - \int_{t_1}^{t_1 + \Delta t} f(q(x_1 + \Delta x, t)) \, dt.
\]

Since \( q \) is essentially constant along each edge, this becomes

\[
\Delta x q_l - \Delta x q_r = \Delta t f(q_l) - \Delta t f(q_r) + O(\Delta t^2),
\]

Taking the limit as \( \Delta t \to 0 \) gives

\[
s(q_r - q_l) = f(q_r) - f(q_l).
\]

Rankine-Hugoniot jump condition

\[
s(q_r - q_l) = f(q_r) - f(q_l).
\]

This must hold for any discontinuity propagating with speed \( s \), even for systems of conservation laws.

For scalar problem, any jump allowed with speed \( s \):

\[
s = \frac{f(q_r) - f(q_l)}{q_r - q_l}.
\]

For systems, \( q_r - q_l \) and \( f(q_r) - f(q_l) \) are vectors, \( s \) scalar,

R-H condition: \( f(q_r) - f(q_l) \) must be scalar multiple of \( q_r - q_l \).

For linear system, \( f(q) = Aq \), this says

\[
A(q_r - q_l) = s(q_r - q_l),
\]

Jump must be an eigenvector, speed \( s \) the eigenvalue.
Figure 11.1 — Shock formation in traffic

Discrete cars: Continuum model: \( f'(q) = u_{\text{max}}(1 - 2q) \)

Figure 11.1 — Shock formation

(a) particle paths (car trajectories) \( u(x, t) = u_{\text{max}}(1 - q(x, t)) \)

Figure 11.1 — Shock formation

(b) characteristics: \( f'(q) = u_{\text{max}}(1 - 2q) \)
**Figure 11.2 — Traffic jam shock wave**

Cars approaching red light \( q_\ell < 1, \ q_r = 1 \)

Shock speed:
\[
s = f(q_r) - f(q_\ell) = \frac{-2u_{\text{max}}q_\ell}{1 - q_\ell} < 0.
\]

**Figure 11.3 — Rarefaction wave**

Cars accelerating at green light \( q_\ell = 1, \ q_r = 0 \)

Characteristic speed \( f'(q) = u_{\text{max}}(1 - 2q) \)

varies from \( f'(q_\ell) = -u_{\text{max}} \) to \( f'(q_r) = u_{\text{max}} \).

**Nonlinear scalar conservation laws**

Burgers’ equation: \( u_t + \left( \frac{1}{2} u^2 \right)_x = 0 \).

Quasilinear form: \( u_t + uu_x = 0 \).

These are equivalent for smooth solutions, not for shocks!

Upwind methods for \( u > 0 \): 
Conservative: \( U^{n+1}_i = U^n_i - \frac{\Delta t}{\Delta x} \left( \frac{1}{2} (U^n_i)^2 - (U^n_{i-1})^2 \right) \)

Quasilinear: \( U^{n+1}_i = U^n_i - \frac{\Delta t}{\Delta x} U^n_i (U^n_i - U^n_{i-1}) \).

Ok for smooth solutions, not for shocks!
Importance of conservation form

Solution to Burgers’ equation using conservative upwind:

Solution to Burgers’ equation using quasilinear upwind:

Weak solutions depend on the conservation law

The conservation laws

\[ u_t + \left( \frac{1}{2} u^2 \right)_x = 0 \]

and

\[ (u^2)_t + \left( \frac{2}{3} u^3 \right)_x = 0 \]

both have the same quasilinear form

\[ u_t + uu_x = 0 \]

but have different weak solutions,

different shock speeds!

Conservation form

The method

\[ Q^{n+1}_i = Q^n_i - \frac{\Delta t}{\Delta x} \left( F_{i+1/2}^n - F_{i-1/2}^n \right) \]

is in conservation form.

The total mass is conserved up to fluxes at the boundaries:

\[ \Delta x \sum_i Q^{n+1}_i = \Delta x \sum_i Q^n_i - \frac{\Delta t}{\Delta x} (F_{+\infty} - F_{-\infty}). \]

Note: an isolated shock must travel at the right speed!
Lax-Wendroff Theorem

Suppose the method is conservative and consistent with
\[ q_t + f(q)_x = 0, \]
\[ F_{i-1/2} = F(Q_{i-1}, Q_i) \quad \text{with} \quad F(q, \bar{q}) = f(q) \]

and Lipschitz continuity of \( F \).

If a sequence of discrete approximations converge to a function \( q(x, t) \) as the grid is refined, then this function is a weak solution of the conservation law.

Note:

Does not guarantee a sequence converges
Two sequences might converge to different weak solutions.
Also need stability and entropy condition.

Non-uniqueness of weak solutions

For scalar problem, any jump allowed with speed:
\[ s = \frac{f(q_r) - f(q_l)}{q_r - q_l}. \]

So even if \( f'(q_r) < f'(q_l) \) the integral form of cons. law is satisfied by a discontinuity propagating at the R-H speed.

In this case there is also a rarefaction wave solution.
In fact, infinitely many weak solutions.
Which one is physically correct?

Vanishing viscosity solution

We want \( q(x, t) \) to be the limit as \( \epsilon \to 0 \) of solution to
\[ q_t + f(q)_x = \epsilon q_{xx}. \]

This selects a unique weak solution:

- Shock if \( f'(q_l) > f'(q_r) \),
- Rarefaction if \( f'(q_l) < f'(q_r) \).

Lax Entropy Condition:

A discontinuity propagating with speed \( s \) in the solution of a convex scalar conservation law is admissible only if \( f'(q_l) > s > f'(q_r) \).