Outline

- Riemann problems and phase plane (on board)
- Non-hyperbolic problems
- Godunov's method for acoustics
- Riemann solvers in Clawpack
- Acoustics in heterogeneous media
- CFL Condition

Reading: Chapters 4 and 5

www.clawpack.org/users Clawpack documentation

Notes:

Non-hyperbolic example

Consider $q_t + Aq_x = 0$ with $q = \begin{bmatrix} u \\ v \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$

Eigenvalues are $\pm i$.

System can be written as:

$$\begin{aligned} u_t + v_x = 0 \qquad &\Longrightarrow u_{tt} = -v_{xt} \\ v_t - u_x = 0 \qquad &\Longrightarrow v_{xt} = u_{xx} \end{aligned}$$

Combining gives $\quad u_{tt} + u_{xx} = 0.$

Laplace's equation: elliptic! Initial value problem ill-posed.

To make well-posed would need to specify boundary conditions at $t = 0$ and $x = a, \ x = b,$ and at final time $t = T$.

Notes:

Fourier analysis of advection equation

Consider advection equation $q_t + \lambda q_x = 0$ with $\lambda \in \mathbb{R}$.

Initial data: single Fourer mode $q(x, 0) = e^{ikx}$.

Then solution has the form

$$q(x,t) = g(t)e^{ikx}.$$

Use

$$\begin{aligned} q_t(x,t) &= g'(t)e^{ikx} \\ q_x(x,t) &= ikg(t)e^{ikx} \end{aligned}$$

PDE gives $g'(t)e^{ikx} + u\left(ikg(t)e^{ikx}\right) = 0$ and hence the ODE:

ODE: $g'(t) = -ik\lambda g(t) \implies$ Solution: $g(t) = e^{-ik\lambda t}$

PDE Solution: $\quad q(x,t) = e^{ikx}e^{-ik\lambda t} = e^{ik(x-\lambda t)} = q(x - \lambda t, 0).$

Notes:

Fourier analysis if λ complex

Consider equation $q_t + \lambda q_x = 0$ with $\lambda = \alpha + i\beta$ with $\beta > 0$.

(A real $\implies$ complex eigenvalues come in conjugate pairs.)

Initial data: single Fourer mode $q(x,0) = e^{ikx}$.

As before, solution is just

$$q(x,t) = e^{-ik\lambda t} e^{ikx}.$$

But now this is:

$$\begin{aligned} q(x,t) &= e^{-ik(\alpha+i\beta)t} e^{ikx} \\ &= e^{k\beta t} e^{ik(x-\alpha t)} \end{aligned}$$

Translates at speed α but also grows exponentially in time.

k can be arbitrarily large $\implies$ ill-posed problem.

Notes:

Finite differences vs. finite volumes

Finite difference Methods

- Pointwise values $Q_i^n \approx q(x_i, t_n)$
- Approximate derivatives by finite differences
- Assumes smoothness

Finite volume Methods

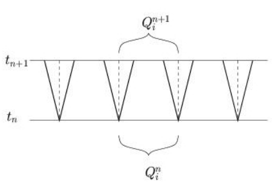
- Approximate cell averages: $Q_i^n \approx \dfrac{1}{\Delta x}\displaystyle\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t_n)\,dx$
- Integral form of conservation law,

$$\frac{\partial}{\partial t}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)\,dx = f(q(x_{i-1/2},t)) - f(q(x_{i+1/2},t))$$

leads to conservation law $q_t + f_x = 0$ but also directly to numerical method.

Notes:

Godunov's Method for $q_t + f(q)_x = 0$

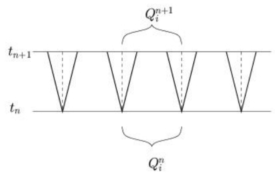


1. Solve Riemann problems at all interfaces, yielding waves $\mathcal{W}_{i-1/2}^p$ and speeds $s_{i-1/2}^p$, for $p = 1,\ 2,\ \ldots,\ m$.

Riemann problem: Original equation with piecewise constant data.

Notes:

Godunov's Method for $q_t + f(q)_x = 0$



Then either:

1. Compute new cell averages by integrating over cell at t_{n+1},
2. Compute fluxes at interfaces and flux-difference:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[F_{i+1/2}^n - F_{i-1/2}^n]$$

3. Update cell averages by contributions from all waves entering cell:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}]$$

where $\mathcal{A}^\pm \Delta Q_{i-1/2} = \sum_{i=1}^{m} (s_{i-1/2}^p)^\pm \mathcal{W}_{i-1/2}^p$.

Notes:

First-order REA Algorithm

1. Reconstruct a piecewise constant function $\tilde{q}^n(x, t_n)$ defined for all x, from the cell averages Q_i^n.

$$\tilde{q}^n(x, t_n) = Q_i^n \quad \text{for all } x \in \mathcal{C}_i.$$

2. Evolve the hyperbolic equation exactly (or approximately) with this initial data to obtain $\tilde{q}^n(x, t_{n+1})$ a time Δt later.

3. Average this function over each grid cell to obtain new cell averages

$$Q_i^{n+1} = \frac{1}{\Delta x}\int_{\mathcal{C}_i} \tilde{q}^n(x, t_{n+1})\, dx.$$

Notes:

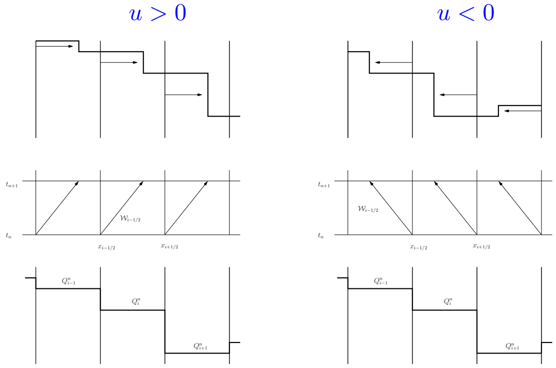
Godunov's method for advection

Q_i^n defines a piecewise constant function

$$\tilde{q}^n(x, t_n) = Q_i^n \text{ for } x_{i-1/2} < x < x_{i+1/2}$$

Discontinuities at cell interfaces $\Longrightarrow$ Riemann problems.

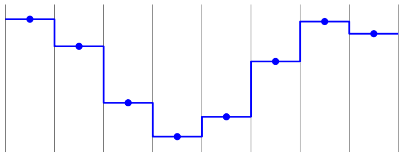
$u > 0$ $u < 0$



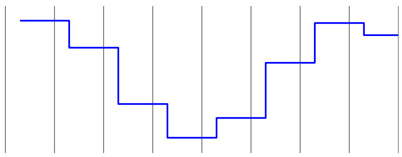
Notes:

First-order REA Algorithm

Cell averages and piecewise constant reconstruction:

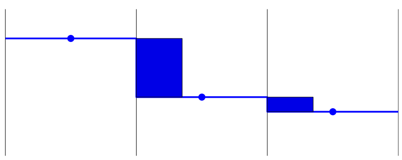


After evolution:



Notes:

Cell update



The cell average is modified by

$$\frac{u\Delta t \cdot (Q_{i-1}^n - Q_i^n)}{\Delta x}$$

So we obtain the upwind method

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n).$$

Notes:

Upwind for advection as a finite volume method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

Advection equation: $f(q) = uq$

$$F_{i-1/2} \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} uq(x_{i-1/2}, t)\, dt.$$

First order upwind:

$$F_{i-1/2} = u^+ Q_{i-1}^n + u^- Q_i^n$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(u^+(Q_i^n - Q_{i-1}^n) + u^-(Q_{i+1}^n - Q_i^n)).$$

where $u^+ = \max(u,0), \quad u^- = \min(u,0).$

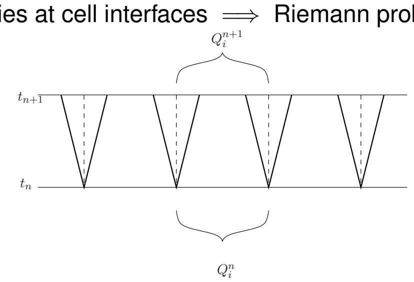
Notes:

Godunov's method

Q_i^n defines a piecewise constant function

$$\tilde{q}^n(x, t_n) = Q_i^n \quad \text{for } x_{i-1/2} < x < x_{i+1/2}$$

Discontinuities at cell interfaces $\Longrightarrow$ Riemann problems.



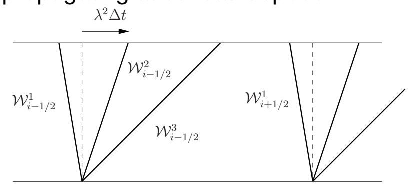
$\tilde{q}^n(x_{i-1/2}, t) \equiv q^{\Downarrow}(Q_{i-1}, Q_i)$ for $t > t_n$.

$$F_{i-1/2}^n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q^{\Downarrow}(Q_{i-1}^n, Q_i^n))\,dt = f(q^{\Downarrow}(Q_{i-1}^n, Q_i^n)).$$

Notes:

Wave-propagation viewpoint

For linear system $q_t + Aq_x = 0$, the Riemann solution consists of waves $\mathcal{W}^p$ propagating at constant speed λ^p.



$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha_{i-1/2}^p r^p \equiv \sum_{p=1}^{m} \mathcal{W}_{i-1/2}^p.$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\lambda^2 \mathcal{W}_{i-1/2}^2 + \lambda^3 \mathcal{W}_{i-1/2}^3 + \lambda^1 \mathcal{W}_{i+1/2}^1\right].$$

Notes:

Upwind wave-propagation algorithm

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\sum_{p=1}^{m} (\lambda^p)^+ \mathcal{W}_{i-1/2}^p + \sum_{p=1}^{m} (\lambda^p)^- \mathcal{W}_{i+1/2}^p\right]$$

or

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right].$$

where the fluctuations are defined by

$$\mathcal{A}^-\Delta Q_{i-1/2} = \sum_{p=1}^{m} (\lambda^p)^- \mathcal{W}_{i-1/2}^p, \quad \text{left-going}$$

$$\mathcal{A}^+\Delta Q_{i-1/2} = \sum_{p=1}^{m} (\lambda^p)^+ \mathcal{W}_{i-1/2}^p, \quad \text{right-going}$$

Notes:

Upwind wave-propagation algorithm

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\sum_{p=1}^{m}(s_{i-1/2}^p)^+\mathcal{W}_{i-1/2}^p + \sum_{p=1}^{m}(s_{i+1/2}^p)^-\mathcal{W}_{i+1/2}^p\right]$$

where

$$s^+ = \max(s,0), \qquad s^- = \min(s,0).$$

Note: Requires only waves and speeds.

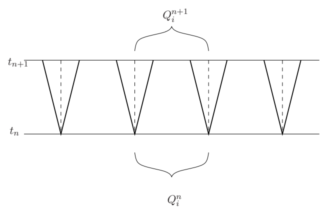
Applicable also to hyperbolic problems not in conservation form.

For $q_t + f(q)_x = 0$, conservative if waves chosen properly,
e.g. using Roe-average of Jacobians.

Great for general software, but only first-order accurate (upwind method for linear systems).

Notes:

Godunov (upwind) on acoustics



Data at time t_n : $\tilde{q}^n(x,t_n) = Q_i^n$ for $x_{i-1/2} < x < x_{i+1/2}$
Solving Riemann problems for small Δt gives solution:

$$\tilde{q}^n(x,t_{n+1}) = \begin{cases} Q_{i-1/2}^* & \text{if } x_{i-1/2} - c\Delta t < x < x_{i-1/2} + c\Delta t, \\ Q_i^n & \text{if } x_{i-1/2} + c\Delta t < x < x_{i+1/2} - c\Delta t, \\ Q_{i+1/2}^* & \text{if } x_{i+1/2} - c\Delta t < x < x_{i+1/2} + c\Delta t, \end{cases}$$

So computing cell average gives:

$$Q_i^{n+1} = \frac{1}{\Delta x}\left[c\Delta t Q_{i-1/2}^* + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t Q_{i+1/2}^*\right].$$

Notes:

Godunov (upwind) on acoustics

$$Q_i^{n+1} = \frac{1}{\Delta x}\left[c\Delta t Q_{i-1/2}^* + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t Q_{i+1/2}^*\right].$$

Solve Riemann problems:

$$Q_i^n - Q_{i-1}^n = \Delta Q_{i-1/2} = \mathcal{W}_{i-1/2}^1 + \mathcal{W}_{i-1/2}^2 = \alpha_{i-1/2}^1 r^1 + \alpha_{i-1/2}^2 r^2,$$
$$Q_{i+1}^n - Q_i^n = \Delta Q_{i+1/2} = \mathcal{W}_{i+1/2}^1 + \mathcal{W}_{i+1/2}^2 = \alpha_{i+1/2}^1 r^1 + \alpha_{i+1/2}^2 r^2,$$

The intermediate states are:

$$Q_{i-1/2}^* = Q_i^n - \mathcal{W}_{i-1/2}^2, \qquad Q_{i+1/2}^* = Q_i^n + \mathcal{W}_{i+1/2}^1,$$

So,

$$Q_i^{n+1} = \frac{1}{\Delta x}\left[c\Delta t(Q_i^n - \mathcal{W}_{i-1/2}^2) + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t(Q_i^n + \mathcal{W}_{i+1/2}^1)\right]$$
$$= Q_i^n - \frac{c\Delta t}{\Delta x}\mathcal{W}_{i-1/2}^2 + \frac{c\Delta t}{\Delta x}\mathcal{W}_{i+1/2}^1.$$

Notes:

Godunov (upwind) on acoustics

Solve Riemann problems:

$$Q_i^n - Q_{i-1}^n = \Delta Q_{i-1/2} = \mathcal{W}_{i-1/2}^1 + \mathcal{W}_{i-1/2}^2 = \alpha_{i-1/2}^1 r^1 + \alpha_{i-1/2}^2 r^2,$$
$$Q_{i+1}^n - Q_i^n = \Delta Q_{i+1/2} = \mathcal{W}_{i+1/2}^1 + \mathcal{W}_{i+1/2}^2 = \alpha_{i+1/2}^1 r^1 + \alpha_{i+1/2}^2 r^2,$$

The waves are determined by solving for α from $R\alpha = \Delta Q$:

$$A = \begin{bmatrix} 0 & K \\ 1/\rho & 0 \end{bmatrix}, \qquad R = \begin{bmatrix} -Z & Z \\ 1 & 1 \end{bmatrix}, \qquad R^{-1} = \frac{1}{2Z}\begin{bmatrix} -1 & Z \\ 1 & Z \end{bmatrix}.$$

So

$$\Delta Q = \begin{bmatrix} \Delta p \\ \Delta u \end{bmatrix} = \alpha^1 \begin{bmatrix} -Z \\ 1 \end{bmatrix} + \alpha^2 \begin{bmatrix} Z \\ 1 \end{bmatrix}$$

with

$$\alpha^1 = \frac{1}{2Z}(-\Delta p + Z\Delta u), \qquad \alpha^2 = \frac{1}{2Z}(\Delta p + Z\Delta u).$$

Notes:

CLAWPACK Riemann solver

The hyperbolic problem is specified by the Riemann solver

- Input: Values of q in each grid cell
- Output: Solution to Riemann problem at each interface.
  - Waves $\mathcal{W}^p \in \mathbb{R}^m$, $p = 1, 2, \ldots, M_w$
  - Speeds $s^p \in \mathbb{R}$, $p = 1, 2, \ldots, M_w$,
  - Fluctuations $\mathcal{A}^-\Delta Q,\ \mathcal{A}^+\Delta Q \in \mathbb{R}^m$

Note: Number of waves M_w often equal to m (length of q), but could be different (e.g. HLL solver has 2 waves).

Fluctuations:

$\mathcal{A}^-\Delta Q =$ Contribution to cell average to left,
$\mathcal{A}^+\Delta Q =$ Contribution to cell average to right

For conservation law, $\mathcal{A}^-\Delta Q + \mathcal{A}^+\Delta Q = f(Q_r) - f(Q_l)$

Notes:

CLAWPACK Riemann solver

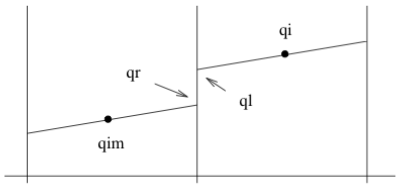
Inputs to `rp1` subroutine:

`ql(i,1:m)` = Value of q at left edge of ith cell,

`qr(i,1:m)` = Value of q at right edge of ith cell,

Warning: The Riemann problem at the interface between cells $i-1$ and i has left state `qr(i-1,:)` and right state `ql(i,:)`.

`rp1` is normally called with `ql = qr = q`, but designed to allow other methods:



Notes:

CLAWPACK Riemann solver

Outputs from `rp1` subroutine:

for system of m equations
with `mw` ranging from 1 to M_w =# of waves

`s(i,mw)` = Speed of wave # `mw` in ith Riemann solution,

`wave(i,1:m,mw)` = Jump across wave # `mw`,

`amdq(i,1:m)` = Left-going fluctuation, updates Q_{i-1}
`apdq(i,1:m)` = Right-going fluctuation, updates Q_i

Notes:

Clawpack acoustics examples

Constant coefficient acoustics:

$CLAW/apps/acoustics/1d/example2/ ... **rp1.f**

Heterogeneous medium with two interfaces:

$IPDE/claw-apps/acoustics-1d-1/ ... **rp1acv.f**

Heterogeneous medium with a single interface:

$CLAW/book/chap9/acoustics/interface/README

Heterogeneous periodic medium:

$CLAW/book/chap9/acoustics/layered/README

Notes:

Coupled advection–acoustics

Flow in pipe with constant background velocity $\bar{u}$.
$\phi(x,t) =$ concentration of advected tracer
$u(x,t),\ p(x,t) =$ acoustic velocity / pressure perturbation

Equations include advection at velocity $\bar{u}$:

$$\begin{aligned}
p_t &+ \bar{u}p_x &+ Ku_x & & = 0 \\
u_t &+ (1/\rho)p_x &+ \bar{u}u_x & & = 0 \\
\phi_t & & &+ \bar{u}\phi_x & = 0
\end{aligned}$$

This is a linear system $q_t + Aq_x = 0$ with

$$q = \begin{bmatrix} p \\ u \\ \phi \end{bmatrix}, \qquad A = \begin{bmatrix} \bar{u} & K & 0 \\ 1/\rho & \bar{u} & 0 \\ 0 & 0 & \bar{u} \end{bmatrix}.$$

Notes:

Coupled advection–acoustics

$$q = \begin{bmatrix} p \\ u \\ \phi \end{bmatrix}, \qquad A = \begin{bmatrix} \bar{u} & K & 0 \\ 1/\rho & \bar{u} & 0 \\ 0 & 0 & \bar{u} \end{bmatrix}.$$

eigenvalues: $\lambda^1 = u - c, \qquad \lambda^2 = u \qquad \lambda^3 = u + c,$

eigenvectors: $r^1 = \begin{bmatrix} -Z \\ 1 \\ 0 \end{bmatrix}, \quad r^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad r^3 = \begin{bmatrix} Z \\ 1 \\ 0 \end{bmatrix},$

where $c = \sqrt{\kappa/\rho}, \;\; Z = \rho c = \sqrt{\rho\kappa}.$

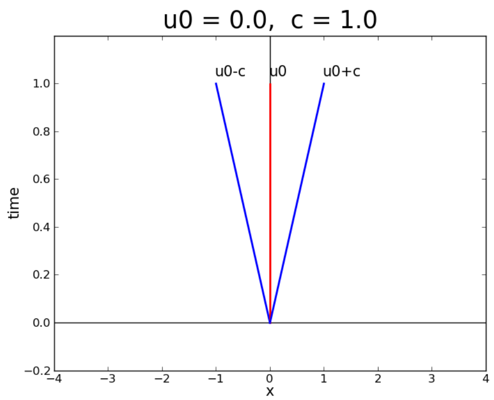
$$R = \begin{bmatrix} -Z & 0 & Z \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad R^{-1} = \frac{1}{2Z}\begin{bmatrix} -1 & Z & 0 \\ 0 & 0 & 1 \\ 1 & Z & 0 \end{bmatrix}.$$

Notes:

Coupled advection–acoustics

Wave structure of solution in the x–t plane

With no advection:

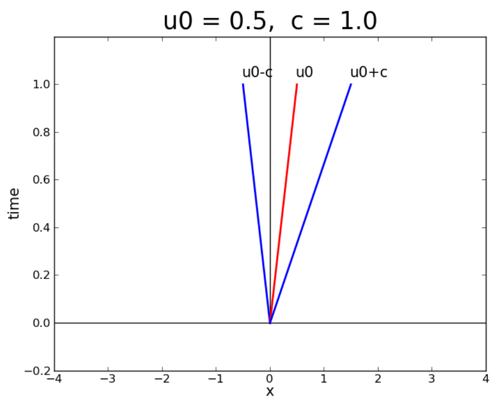


Notes:

Coupled advection–acoustics

Wave structure of solution in the x–t plane

Subsonic case ($|u_0| < c$):

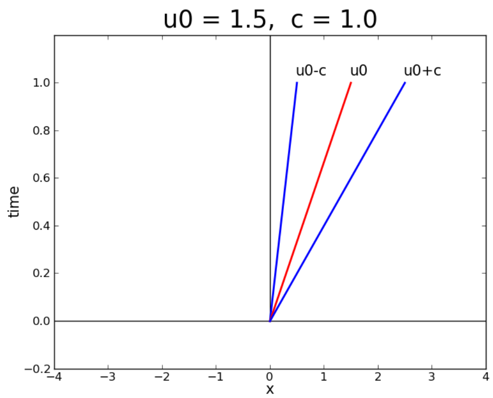


Notes:

Coupled advection–acoustics

Wave structure of solution in the x–t plane

Supersonic case ($|u_0| > c$):



Notes:

Wave propagation in heterogeneous medium

Linear system $q_t + A(x)q_x = 0$. For acoustics:

$$A = \begin{bmatrix} 0 & K(x) \\ 1/\rho(x) & 0 \end{bmatrix}.$$

eigenvalues: $\lambda^1 = -c(x), \qquad \lambda^2 = +c(x),$

where $c(x) = \sqrt{\kappa(x)/\rho(x)}$ = local speed of sound.

eigenvectors: $r^1(x) = \begin{bmatrix} -Z(x) \\ 1 \end{bmatrix}, \qquad r^2(x) = \begin{bmatrix} Z(x) \\ 1 \end{bmatrix}$

where $Z(x) = \rho c = \sqrt{\rho\kappa}$ = impedance.

$$R(x) = \begin{bmatrix} -Z(x) & Z(x) \\ 1 & 1 \end{bmatrix}, \qquad R^{-1}(x) = \frac{1}{2Z(x)} \begin{bmatrix} -1 & Z(x) \\ 1 & Z(x) \end{bmatrix}.$$

Cannot diagonalize unless $Z(x)$ is constant.

Notes:

Wave propagation in heterogeneous medium

Multiply system

$$q_t + A(x)q_x = 0$$

by $R^{-1}(x)$ on left to obtain

$$R^{-1}(x)q_t + R^{-1}(x)A(x)R(x)\,R^{-1}(x)q_x = 0$$

or

$$(R^{-1}(x)q)_t + \Lambda(x)\,\left[(R^{-1}(x)q)_x - R_x^{-1}(x)q\right] = 0$$

Let $w(x,t) = R^{-1}(x)q(x,t)$ (characteristic variable).
There is a coupling term on the right:

$$w_t + \Lambda(x)\,w_x = \Lambda(x)R_x^{-1}(x)R(x)w$$

$\Longrightarrow$ reflections (unless $R_x^{-1}(x) \equiv 0$).

Notes:

Wave propagation in heterogeneous medium

Generalized Riemann problem: single jump discontinuity in $q(x,0)$ and in $K(x)$ and $\rho(x)$.

Decompose jump in q as linear combination of eigenvectors, with

- left-going waves: eigenvectors for material on left,
- right-going waves: eigenvectors for material on right.

$$R(x) = \begin{bmatrix} -Z(x) & Z(x) \\ 1 & 1 \end{bmatrix}, \qquad R^{-1}(x) = \frac{1}{2Z(x)} \begin{bmatrix} -1 & Z(x) \\ 1 & Z(x) \end{bmatrix}.$$

Riemann solution: decompose

$$q_r - q_l = \alpha^1 \begin{bmatrix} -Z_l \\ 1 \end{bmatrix} + \alpha^2 \begin{bmatrix} Z_r \\ 1 \end{bmatrix} = \mathcal{W}^1 + \mathcal{W}^2$$

The waves propagate with speeds $s^1 = -c_l$ and $s^2 = c_r$.

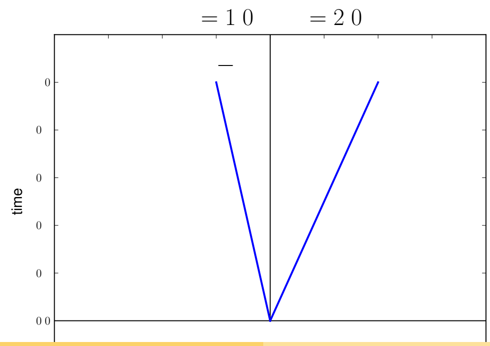
Notes:

Wave propagation in heterogeneous medium

Riemann solution: decompose

$$q_r - q_l = \alpha^1 \begin{bmatrix} -Z_l \\ 1 \end{bmatrix} + \alpha^2 \begin{bmatrix} Z_r \\ 1 \end{bmatrix} = \mathcal{W}^1 + \mathcal{W}^2$$

The waves propagate with speeds $s^1 = -c_l$ and $s^2 = c_r$.



Notes:

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Heterogeneous medium with a single interface:

$CLAW/book/chap9/acoustics/interface/README

Heterogeneous periodic medium:

$CLAW/book/chap9/acoustics/layered/README

Notes:

The CFL Condition

Domain of dependence: The solution $q(X,T)$ depends on the data $q(x,0)$ over some set of x values, $x \in \mathcal{D}(X,T)$.

Advection: $q(X,T) = q(X-uT,0)$ and so $\mathcal{D}(X,T) = \{X-uT\}$.

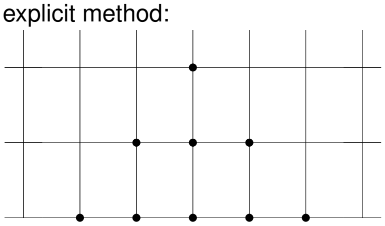
The CFL Condition: A numerical method can be convergent only if its numerical domain of dependence contains the true domain of dependence of the PDE, at least in the limit as Δt and Δx go to zero.

Note: Necessary but not sufficient for stability!

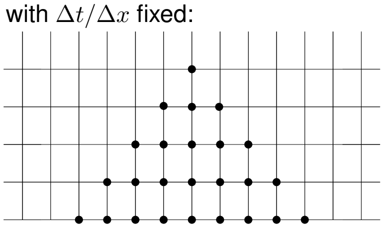
Notes:

Numerical domain of dependence

With a 3-point explicit method:



On a finer grid with $\Delta t/\Delta x$ fixed:



Notes:

The CFL Condition

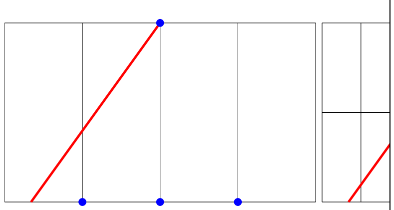
For the method to be stable, the numerical domain of dependence must include the true domain of dependence.

For advection, the solution is constant along characteristics,

$$q(x,t) = q(x-ut,0)$$

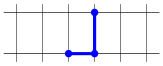
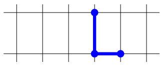
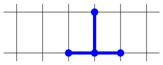
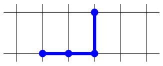
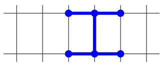
For a 3-point method, CFL condition requires $\left|\frac{u\Delta t}{\Delta x}\right| \le 1$.

If this is violated:
True solution is determined by data at a point $x-ut$ that is ignored by the numerical method, even as the grid is refined.



Notes:

Stencil / CFL Condition

Stencil	CFL Condition
	$0 \le \frac{u\Delta t}{\Delta x} \le 1$
	$-1 \le \frac{u\Delta t}{\Delta x} \le 0$
	$-1 \le \frac{u\Delta t}{\Delta x} \le 1$
	$0 \le \frac{u\Delta t}{\Delta x} \le 2$
	$-\infty < \frac{u\Delta t}{\Delta x} < \infty$

Notes:

R.J. LeVeque, University of Washington IPDE 2011, June 24, 2011 [FVMHP Sec. 4.4]

Linear hyperbolic systems

Linear system of m equations: $q(x,t) \in \mathbb{R}^m$ for each (x,t) and

$$q_t + Aq_x = 0, \qquad -\infty < x, \infty, \;\; t \ge 0.$$

A is $m \times m$ with eigenvalues λ^p and eigenvectors r^p, for $p = 1,\; 2,\; \ldots,\; m$:

$$Ar^p = \lambda^p r^p.$$

Combining these for $p = 1,\; 2,\; \ldots,\; m$ gives

$$AR = R\Lambda$$

where

$$R = [r^1 \;\; r^2 \;\; \cdots \;\; r^m], \qquad \Lambda = \text{diag}(\lambda^1,\; \lambda^2,\; \ldots,\; \lambda^m).$$

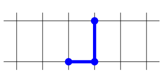
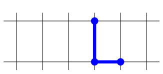
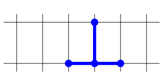
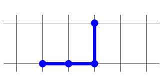
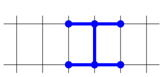
The system is hyperbolic if the eigenvalues are real and R is invertible. Then A can be diagonalized:

$$R^{-1}AR = \Lambda$$

Notes:

R.J. LeVeque, University of Washington IPDE 2011, June 24, 2011 [FVMHP Chap. 3]

Stencil / CFL Condition

Stencil	CFL Condition
	$0 \le \frac{\lambda_p\Delta t}{\Delta x} \le 1, \quad \forall p$
	$-1 \le \frac{\lambda_p\Delta t}{\Delta x} \le 0, \quad \forall p$
	$-1 \le \frac{\lambda_p\Delta t}{\Delta x} \le 1, \quad \forall p$
	$0 \le \frac{\lambda_p\Delta t}{\Delta x} \le 2, \quad \forall p$
	$-\infty < \frac{\lambda_p\Delta t}{\Delta x} < \infty, \quad \forall p$

Notes:

R.J. LeVeque, University of Washington IPDE 2011, June 24, 2011 [FVMHP Sec. 4.4]