Chapter 10 Exercises

From: *Finite Difference Methods for Ordinary and Partial Differential Equations*
by R. J. LeVeque, SIAM, 2007. `http://www.amath.washington.edu/~rjl/fdmbook`

Exercise 10.1 *(One-sided and centered methods)*

Let $U = [U_0,\ U_1,\ \ldots,\ U_m]^T$ be a vector of function values at equally spaced points on the interval $0 \le x \le 1$, and suppose the underlying function is periodic and smooth. Then we can approximate the first derivative u_x at all of these points by DU, where D is circulant matrix such as

$$D_- = \frac{1}{h}\begin{bmatrix} 1 & & & & -1 \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{bmatrix}, \qquad D_+ = \frac{1}{h}\begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \\ 1 & & & & -1 \end{bmatrix} \tag{E10.1a}$$

for first-order accurate one-sided approximations or

$$D_0 = \frac{1}{2h}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix} \tag{E10.1b}$$

for a second-order accurate centered approximation. (These are illustrated for a grid with $m + 1 = 5$ unknowns and $h = 1/5$.)

The advection equation $u_t + au_x = 0$ on the interval $0 \le x \le 1$ with periodic boundary conditions gives rise to the MOL discretization $U'(t) = -aDU(t)$ where D is one of the matrices above.

(a) Discretizing $U' = -aD_-U$ by forward Euler gives the first order upwind method

$$U_j^{n+1} = U_j^n - \frac{ak}{h}(U_j^n - U_{j-1}^n), \tag{E10.1c}$$

where the index i runs from 0 to m with addition of indices performed mod $m + 1$ to incorporate the periodic boundary conditions.

Suppose instead we discretize the MOL equation by the second-order Taylor series method,

$$U^{n+1} = U^n - akD_-U^n + \frac{1}{2}(ak)^2 D_-^2 U^n. \tag{E10.1d}$$

Compute D_-^2 and also write out the formula for U_j^n that results from this method.

(b) How accurate is the method derived in part (a) compared to the Beam-Warming method, which is also a 3-point one-sided method?

(c) Suppose we make the method (E10.1c) more symmetric:

$$U^{n+1} = U^n - \frac{ak}{2}(D_+ + D_-)U^n + \frac{1}{2}(ak)^2 D_+ D_- U^n. \tag{E10.1e}$$

Write out the formula for U_j^n that results from this method. What standard method is this?

Exercise 10.2 *(Eigenvalues of A_ϵ for upwind)*

(a) Produce a plot similar to those shown in Figure 10.1 for the upwind method (10.21) with the same values of $a = 1$, $h = 1/50$ and $k = 0.8h$ used in that figure.

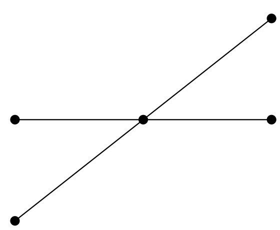
(b) Produce the corresponding plot if the one-sided method (10.22) is instead used with the same values of a, h, and k.

Exercise 10.3 *(skewed leapfrog)*

Suppose $a > 0$ and consider the following *skewed leapfrog* method for solving the advection equation $u_t + au_x = 0$:

$$U_j^{n+1} = U_{j-2}^{n-1} - \left(\frac{ak}{h} - 1\right)(U_j^n - U_{j-2}^n). \tag{E10.3a}$$

The stencil of this method is



Note that if $ak/h \approx 1$ then this stencil roughly follows the characteristic of the advection equation and might be expected to be more accurate than standard leapfrog. (If $ak/h = 1$ the method is exact.)

(a) What is the order of accuracy of this method?

(b) For what range of Courant number ak/h does this method satisfy the CFL condition?

(c) Show that the method is in fact stable for this range of Courant numbers by doing von Neumann analysis. **Hint:** Let $\gamma(\xi) = e^{i\xi h} g(\xi)$ and show that γ satisfies a quadratic equation closely related to the equation (10.34) that arises from a von Neumann analysis of the leapfrog method.

Exercise 10.4 *(trapezoid method for advection)*

Consider the method

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}(U_j^n - U_{j-1}^n + U_j^{n+1} - U_{j-1}^{n+1}). \tag{E10.4a}$$

for the advection equation $u_t + au_x = 0$ on $0 \leq x \leq 1$ with periodic boundary conditions.

(a) This method can be viewed as the trapezoidal method applied to an ODE system $U'(t) = AU(t)$ arising from a method of lines discretization of the advection equation. What is the matrix A? Don't forget the boundary conditions.

(b) Suppose we want to fix the Courant number ak/h as $k,\ h \to 0$. For what range of Courant numbers will the method be stable if $a > 0$? If $a < 0$? Justify your answers in terms of eigenvalues of the matrix A from part (a) and the stability regions of the trapezoidal method.

(c) Apply von Neumann stability analysis to the method (E10.4a). What is the amplification factor $g(\xi)$?

(d) For what range of ak/h will the CFL condition be satisfied for this method (with periodic boundary conditions)?

(e) Suppose we use the same method (E10.4a) for the initial-boundary value problem with $u(0,t) = g_0(t)$ specified. Since the method has a one-sided stencil, no numerical boundary condition is needed at the right boundary (the formula (E10.4a) can be applied at x_{m+1}). For what range of ak/h will the CFL condition be satisfied in this case? What are the eigenvalues of the A matrix for this case and when will the method be stable?

Exercise 10.5 *(modified equation for Lax-Wendroff)*

Derive the modified equation (10.45) for the Lax-Wendroff method.

Exercise 10.6 *(modified equation for Beam-Warming)*

Show that the Beam-Warming method (10.26) is second order accurate on the advection equation and also derive the modified equation (10.47) on which it is third order accurate.

Exercise 10.7 *(modified equation for trapezoidal)*

Determine the modified equation on which the method

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}(U_j^n - U_{j-1}^n + U_j^{n+1} - U_{j-1}^{n+1}).$$

from Exercise 10.4 is second order accurate. Is this method predominantly dispersive or dissipative?

Exercise 10.8 *(computing with Lax-Wendroff and upwind)*

The m-file `advection_LW_pbc.m` implements the Lax-Wendroff method for the advection equation on $0 \leq x \leq 1$ with periodic boundary conditions.

(a) Observe how this behaves with $m+1 = 50,\ 100,\ 200$ grid points. Change the final time to `tfinal = 0.1` and use the m-files `error_table.m` and `error_loglog.m` to verify second order accuracy.

(b) Modify the m-file to create a version `advection_up_pbc.m` implementing the upwind method and verify that this is first order accurate.

(c) Keep m fixed and observe what happens with `advection_up_pbc.m` if the time step k is reduced, e.g. try $k = 0.4h$, $k = 0.2h$, $k = 0.1h$. When a convergent method is applied to an ODE we expect better accuracy as the time step is reduced and we can view the upwind method as an ODE solver applied to an MOL system. However, you should observe decreased accuracy as $k \to 0$ with h fixed. Explain this apparent paradox. **Hint:** What ODE system are we solving more accuracy? You might also consider the modified equation (10.44).

Exercise 10.9 *(computing with leapfrog)*

The m-file `advection_LW_pbc.m` implements the Lax-Wendroff method for the advection equation on $0 \le x \le 1$ with periodic boundary conditions.

(a) Modify the m-file to create a version `advection_lf_pbc.m` implementing the leapfrog method and verify that this is second order accurate. Note that you will have to specify two levels of initial data. For the convergence test set $U_j^1 = u(x_j, k)$, the true solution at time k.

(b) Modify `advection_lf_pbc.m` so that the initial data consists of a wave packet

$$\eta(x) = \exp(-\beta(x-0.5)^2)\sin(\xi x) \tag{E10.9a}$$

Work out the true solution $u(x,t)$ for this data. Using $\beta = 100$, $\xi = 80$ and $U_j^1 = u(x_j, k)$, test that your code still exhibits second order accuracy for k and h sufficiently small.

(c) Using $\beta = 100$, $\xi = 150$ and $U_j^1 = u(x_j, k)$, estimate the group velocity of the wave packet computed with leapfrog using $m = 199$ and $k = 0.4h$. How well does this compare with the value (10.52) predicted by the modified equation?

Exercise 10.10 *(Lax-Richtmyer stability of leapfrog as a one-step method)*

Consider the leapfrog method for the advection equation $u_t + au_x = 0$ on $0 \le x \le 1$ with periodic boundary conditions. From the von Neumann analysis of Example 10.4 we expect this method to be stable for $|ak/h| < 1$. However, the Lax Equivalence theorem as stated in Section 9.5 only applies to 1-step (2-level) methods. The point of this exercise is to show that the 3-level leapfrog method can be interpreted as a 1-step method to which the Lax Equivalence theorem applies.

The leapfrog method $U^{n+1} = U^{n-1} + 2kAU^n$ can be rewritten as

$$\begin{bmatrix} U^{n+1} \\ U^n \end{bmatrix} = \begin{bmatrix} 2kA & I \\ I & 0 \end{bmatrix} \begin{bmatrix} U^n \\ U^{n-1} \end{bmatrix}, \tag{E10.10a}$$

which has the form $V^{n+1} = BV^n$.

(a) Show that the matrix B defined by (E10.10a) has $2(m+1)$ eigenvectors of the form

$$\begin{bmatrix} g_p^- u^p \\ u^p \end{bmatrix}, \quad \begin{bmatrix} g_p^+ u^p \\ u^p \end{bmatrix}, \quad \text{for } p = 1,\ 2,\ \ldots,\ m+1, \tag{E10.10b}$$

where $u^p \in \mathbb{R}^{m+1}$ are the eigenvectors of A given by (10.12) and $g_p^\pm$ are the two roots of a quadratic equation. Explain how this quadratic equation relates to (10.34) (what values of ξ are relevant for this grid?)

What are the eigenvalues of B?

(b) Show that if

$$|ak/h| < 1 \tag{E10.10c}$$

then the eigenvalues of B are distinct with magnitude equal to 1.

(c) The result of part (b) is not sufficient to prove that leapfrog is Lax-Ricthmyer stable. The matrix B is not normal and the matrix of right eigenvectors R with columns given by (E10.10b) is not unitary. By (15.8) we have

$$\|B^n\|_2 \le \|R\|_2 \|R^{-1}\|_2 = \kappa_2(R). \tag{E10.10d}$$

To prove uniform power boundedness and stability we must show that the condition number of R is uniformly bounded as $k \to 0$ provided (E10.10c) is satisfied.

Prove this by the following steps:

(i) Let

$$U = \frac{1}{\sqrt{m+1}} \begin{bmatrix} u^1 & u^2 & \cdots & u^p \end{bmatrix} \in \mathbb{R}^{(m+1)\times(m+1)} \tag{E10.10e}$$

be an appropriately scaled right eigenvector matrix of A. Show that with this scaling, U is a unitary matrix: $U^H U = I$.

(ii) Show that the right eigenvector matrix of B can be written as

$$R = \begin{bmatrix} UG^- & UG^+ \\ U & U \end{bmatrix} \tag{E10.10f}$$

where $G^\pm = \operatorname{diag}(g_1^\pm,\ \ldots,\ g_{m+1}^\pm)$.

(iii) Show that if $x = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{2(m+1)}$ has $\|z\|_2 = 1$ then $\|Rz\|_2 \le C$ for some constant independent of m, and hence $\|R\|_2 \le C$ for all k. (It is fairly easy to show this with $C = 2\sqrt{2}$ and with a bit more work that in fact $\|R\|_2 = 2$ for all k.)

(iv) Let

$$L = \begin{bmatrix} G^- U^H & U^H \\ G^+ U^H & U^H \end{bmatrix}. \tag{E10.10g}$$

Show that LR is a diagonal matrix and hence R^{-1} is a diagonal scaling of the matrix L. Determine R^{-1}.

(v) Use the previous result to show that

$$\|R^{-1}\|_2 \le \frac{C}{1-\nu^2} \tag{E10.10h}$$

for some constant C, where $\nu = ak/h$ is the Courant number.

(vi) Conclude from the above steps that B is uniformly power bounded and hence the leapfrog method is Lax-Richtmyer stable provided that $|\nu| < 1$.

(d) Show that the leapfrog method with periodic boundary conditions is also stable in the case $|ak/h| = 1$ if $m+1$ is not divisible by 4. Find a good set of initial data U^0 and U^1 to illustrate the instability that arises if $m+1$ is divisible by 4 and perform a calculation that demonstrates nonconvergence in this case.

Exercise 10.11 *(Modified equation for Gauss-Seidel)*

Exercise 9.4 illustrates how the Jacobi iteration for solving the boundary value problem $u_{xx}(x) = f(x)$ can be viewed as an explicit time-stepping method for the heat equation $u_t(x,t) = u_{xx}(x,t) - f(x)$ with a time step $k = h^2/2$.

Now consider the Gauss-Seidel method for solving the linear system,

$$U_j^{n+1} = \frac{1}{2}(U_{j-1}^{n+1} + U_{j+1}^{n} - h^2 f(x_j)). \tag{E10.11a}$$

This can be viewed as a time stepping method for some PDE. Compute the modified equation for this finite difference method and determine what PDE it is consistent with if we let $k = h^2/2$ again. Comment on how this relates to the observation in Section 4.2.1 that Gauss-Seidel takes roughly half as many iterations as Jacobi to converge.