

AMath 574 March 9, 2011

Today:

- Approximate Riemann solvers
- Multidimensional

Friday:

- AMR

Projects: Make an appointment this week, and see

<http://www.clawpack.org/links/burgersadv>

Notes:

Shallow water equations

$h(x, t)$ = depth

$u(x, t)$ = velocity (depth averaged, varies only with x)

Conservation of mass and momentum hu gives system of two equations.

mass flux = hu ,

momentum flux = $(hu)u + p$ where p = hydrostatic pressure

$$\begin{aligned} h_t + (hu)_x &= 0 \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2 \right)_x &= 0 \end{aligned}$$

Jacobian matrix:

$$f'(q) = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix}, \quad \lambda = u \pm \sqrt{gh}.$$

Notes:

Roe solver for Shallow Water

Given h_l, u_l, h_r, u_r , define

$$\bar{h} = \frac{h_l + h_r}{2}, \quad \hat{u} = \frac{\sqrt{h_l}u_l + \sqrt{h_r}u_r}{\sqrt{h_l} + \sqrt{h_r}}$$

Then

\hat{A} = Jacobian matrix evaluated at this average state

satisfies

$$A(q_r - q_l) = f(q_r) - f(q_l).$$

- Roe condition is satisfied,
- Isolated shock modeled well,
- Wave propagation algorithm is conservative,
- High resolution methods obtained using corrections with limited waves.

Notes:

Roe solver for Shallow Water

Given h_l, u_l, h_r, u_r , define

$$\bar{h} = \frac{h_l + h_r}{2}, \quad \hat{u} = \frac{\sqrt{h_l}u_l + \sqrt{h_r}u_r}{\sqrt{h_l} + \sqrt{h_r}}$$

Eigenvalues of $\hat{A} = f'(\hat{q})$ are:

$$\hat{\lambda}^1 = \hat{u} - \hat{c}, \quad \hat{\lambda}^2 = \hat{u} + \hat{c}, \quad \hat{c} = \sqrt{g\bar{h}}.$$

Eigenvectors:

$$\hat{r}^1 = \begin{bmatrix} 1 \\ \hat{u} - \hat{c} \end{bmatrix}, \quad \hat{r}^2 = \begin{bmatrix} 1 \\ \hat{u} + \hat{c} \end{bmatrix}.$$

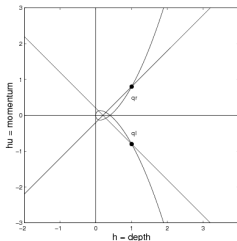
Examples in **Clawpack 4.3** to be converted soon!

Notes:

Potential failure of linearized solvers

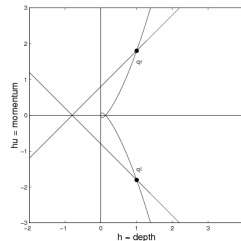
Consider shallow water with $h_l = h_r$ and $u_r = -u_l \gg 1$.

Outflow away from interface \implies small intermediate h_m .



With $u_r = 0.8$

Roe $h_m > 0$



With $u_r = 1.8$

Roe $h_m < 0$

Notes:

HLL Solver

Harten – Lax – van Leer (1983): Use only 2 waves with
 s^1 = minimum characteristic speed
 s^2 = maximum characteristic speed

$$\mathcal{W}^1 = Q^* - Q_\ell, \quad \mathcal{W}^2 = Q_r - Q^*$$

Conservation implies unique value for middle state Q^* :

$$s^1 \mathcal{W}^1 + s^2 \mathcal{W}^2 = f(Q_r) - f(Q_\ell)$$

$$\implies Q^* = \frac{f(Q_r) - f(Q_\ell) - s^2 Q_r + s^1 Q_\ell}{s^1 - s^2}.$$

Choice of speeds:

- Max and min of expected speeds over entire problem,
- Max and min of eigenvalues of $f'(Q_\ell)$ and $f'(Q_r)$.

Notes:

HLLC Solver

Einfeldt: Choice of speeds for gas dynamics (or shallow water) that **guarantees positivity**.

Based on characteristic speeds and Roe averages:

$$s_{i-1/2}^1 = \min_p(\min(\lambda_i^p, \hat{\lambda}_{i-1/2}^p)),$$

$$s_{i-1/2}^2 = \max_p(\max(\lambda_{i+1}^p, \hat{\lambda}_{i-1/2}^p)).$$

where

λ_i^p is the p th eigenvalue of the Jacobian $f'(Q_i)$,

$\hat{\lambda}_{i-1/2}^p$ is the p th eigenvalue using Roe average $f'(\hat{Q}_{i-1/2})$

Notes:

Wave propagation methods

- Solving Riemann problem gives waves $\mathcal{W}_{i-1/2}^p$,

$$Q_i - Q_{i-1} = \sum_p \mathcal{W}_{i-1/2}^p$$

and speeds $s_{i-1/2}^p$. (Usually approximate solver used.)

- These waves update neighboring cell averages depending on sign of s^p (Godunov's method) via fluctuations.
- Waves also give characteristic decomposition of slopes:

$$q_x(x_{i-1/2}, t) \approx \frac{Q_i - Q_{i-1}}{\Delta x} = \frac{1}{\Delta x} \sum_p \mathcal{W}_{i-1/2}^p$$

- Apply limiter to each wave to obtain $\tilde{\mathcal{W}}_{i-1/2}^p$.
- Use limited waves in second-order correction terms.

Notes:

High-resolution wave-propagation algorithm

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} (\mathcal{A}^- \Delta Q_{i+1/2} + \mathcal{A}^+ \Delta Q_{i-1/2}) - \frac{\Delta t}{\Delta x} (\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}),$$

where

$$\tilde{F}_{i-1/2} = \frac{1}{2} \sum_{p=1}^m |s_{i-1/2}^p| \left(1 - \frac{\Delta t}{\Delta x} |s_{i-1/2}^p| \right) \tilde{\mathcal{W}}_{i-1/2}^p.$$

$\tilde{\mathcal{W}}_{i-1/2}^p$ represents a limited version of the wave $\mathcal{W}_{i-1/2}^p$, obtained by comparing this jump with the jump $\mathcal{W}_{I-1/2}^p$ in the same family at the neighboring Riemann problem in the upwind direction,

$$I = \begin{cases} i-1 & \text{if } s_{i-1/2}^p > 0 \\ i+1 & \text{if } s_{i-1/2}^p < 0. \end{cases}$$

Notes:

Wave limiters

Let $\mathcal{W}_{i-1/2} = Q_i^n - Q_{i-1}^n$.

Upwind: $Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x} \mathcal{W}_{i-1/2}$.

Lax-Wendroff:

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x} \mathcal{W}_{i-1/2} - \frac{\Delta t}{\Delta x} (\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2})$$

$$\tilde{F}_{i-1/2} = \frac{1}{2} \left(1 - \left| \frac{u\Delta t}{\Delta x} \right| \right) |u| \mathcal{W}_{i-1/2}$$

High-resolution method:

$$\tilde{F}_{i-1/2} = \frac{1}{2} \left(1 - \left| \frac{u\Delta t}{\Delta x} \right| \right) |u| \tilde{\mathcal{W}}_{i-1/2}$$

where $\tilde{\mathcal{W}}_{i-1/2} = \Phi_{i-1/2} \mathcal{W}_{i-1/2}$.

Notes:

Extension to linear systems

Approach 1: Diagonalize the system to

$$v_t + \Lambda v_x = 0$$

Apply scalar algorithm to each component.

Approach 2:

Solve the linear Riemann problem to decompose $Q_i^n - Q_{i-1}^n$ into waves.

Apply a wave limiter to each wave.

These are equivalent.

Important to apply limiters to waves or characteristic components, rather than to original variables.

Notes:

Wave limiters for system

$Q_i - Q_{i-1}$ is split into waves $\mathcal{W}_{i-1/2}^p = \alpha_{i-1/2}^p r_{i-1/2}^p \in \mathbb{R}^m$.

Replace by $\tilde{\mathcal{W}}_{i-1/2}^p = \Phi(\theta_{i-1/2}^p) \mathcal{W}_{i-1/2}^p$ where

$$\theta_{i-1/2}^p = \frac{\mathcal{W}_{i-1/2}^p \cdot \mathcal{W}_{I-1/2}^p}{\mathcal{W}_{i-1/2}^p \cdot \mathcal{W}_{i-1/2}^p} = \frac{\alpha_{I-1/2}^p}{\alpha_{i-1/2}^p} \text{ if } r_{i-1/2}^p = r_{I-1/2}^p$$

where

$$I = \begin{cases} i-1 & \text{if } s_{i-1/2}^p > 0 \\ i+1 & \text{if } s_{i-1/2}^p < 0. \end{cases}$$

In the scalar case this reduces to

$$\theta_{i-1/2}^1 = \frac{\mathcal{W}_{I-1/2}^1}{\mathcal{W}_{i-1/2}^1} = \frac{Q_I - Q_{I-1}}{Q_i - Q_{i-1}}$$

Notes:

First order hyperbolic PDE in 2 space dimensions

Advection equation: $q_t + uq_x + vq_y = 0$

First-order system: $q_t + Aq_x + Bq_y = 0$

where $q \in \mathbb{R}^m$ and $A, B \in \mathbb{R}^{m \times m}$.

Hyperbolic if $\cos(\theta)A + \sin(\theta)B$ is diagonalizable with real eigenvalues, for all angles θ .

This is required so that plane-wave data gives a 1d hyperbolic problem:

$$q(x, y, 0) = \breve{q}(x \cos \theta + y \sin \theta) \quad (\breve{q})$$

implies contours of q in x - y plane are orthogonal to θ -direction.

Notes:

Acoustics in 2 dimensions

$$p_t + K_0(u_x + v_y) = 0$$

$$\rho_0 u_t + p_x = 0$$

$$\rho_0 v_t + p_y = 0$$

$$A = \begin{bmatrix} 0 & K_0 & 0 \\ 1/\rho_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R^x = \begin{bmatrix} -Z_0 & 0 & Z_0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Solving $q_t + Aq_x = 0$ gives pressure waves in (p, u) .
 x -variations in v are stationary.

$$B = \begin{bmatrix} 0 & 0 & K_0 \\ 0 & 0 & 0 \\ 1/\rho_0 & 0 & 0 \end{bmatrix} \quad R^y = \begin{bmatrix} -Z_0 & 0 & Z_0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Solving $q_t + Bq_y = 0$ gives pressure waves in (p, v) .
 y -variations in u are stationary.

Notes:

2d finite volume method

$$Q_{ij}^{n+1} = Q_{ij}^n - \frac{\Delta t}{\Delta x} [F_{i+1/2,j}^n - F_{i-1/2,j}^n] - \frac{\Delta t}{\Delta y} [G_{i,j+1/2}^n - G_{i,j-1/2}^n].$$

Fluctuation form:

$$Q_{ij}^{n+1} = Q_{ij}^n - \frac{\Delta t}{\Delta x} (A^+ \Delta Q_{i-1/2,j} + A^- \Delta Q_{i+1/2,j}) - \frac{\Delta t}{\Delta y} (B^+ \Delta Q_{i,j-1/2} + B^- \Delta Q_{i,j+1/2}) - \frac{\Delta t}{\Delta x} (\tilde{F}_{i+1/2,j} - \tilde{F}_{i-1/2,j}) - \frac{\Delta t}{\Delta y} (\tilde{G}_{i,j+1/2} - \tilde{G}_{i,j-1/2}).$$

The \tilde{F} and \tilde{G} are **correction fluxes** to go beyond Godunov's upwind method.

Incorporate approximations to second derivative terms in each direction (q_{xx} and q_{yy}) and mixed term q_{xy} .

Notes:

Wave propagation algorithms in 2D

Clawpack requires:

Normal Riemann solver `rpn2.f`

Solves 1d Riemann problem $q_t + Aq_x = 0$

Decomposes $\Delta Q = Q_{ij} - Q_{i-1,j}$ into $\mathcal{A}^+ \Delta Q$ and $\mathcal{A}^- \Delta Q$.

For $q_t + Aq_x + Bq_y = 0$, split using eigenvalues, vectors:

$$A = R\Lambda R^{-1} \implies A^- = R\Lambda^- R^{-1}, A^+ = R\Lambda^+ R^{-1}$$

Input parameter `ixy` determines if it's in x or y direction.

In latter case splitting is done using B instead of A .

This is all that's required for dimensional splitting.

Transverse Riemann solver `rpt2.f`

Decomposes $\mathcal{A}^+ \Delta Q$ into $B^- \mathcal{A}^+ \Delta Q$ and $B^+ \mathcal{A}^+ \Delta Q$ by splitting this vector into eigenvectors of B .

(Or splits vector into eigenvectors of A if `ixy=2`.)

Notes:

Acoustics in heterogeneous media

$$q_t + A(x,y)q_x + B(x,y)q_y = 0, \quad q = (p, u, v)^T,$$

where

$$A = \begin{bmatrix} 0 & K(x,y) & 0 \\ 1/\rho(x,y) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & K(x,y) \\ 0 & 0 & 0 \\ 1/\rho(x,y) & 0 & 0 \end{bmatrix}.$$

Note: Not in conservation form!

Wave propagation still makes sense. In x -direction:

$$\mathcal{W}^1 = \alpha^1 \begin{bmatrix} -Z_{i-1,j} \\ 1 \\ 0 \end{bmatrix}, \quad \mathcal{W}^2 = \alpha^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathcal{W}^3 = \alpha^3 \begin{bmatrix} Z_{ij} \\ 1 \\ 0 \end{bmatrix}.$$

Wave speeds: $s_{i-1/2,j}^1 = -c_{i-1,j}$, $s_{i-1/2,j}^2 = 0$, $s_{i-1/2,j}^3 = +c_{ij}$.

Acoustics in heterogeneous media

$$\mathcal{W}^1 = \alpha^1 \begin{bmatrix} -Z_{i-1,j} \\ 1 \\ 0 \end{bmatrix}, \quad \mathcal{W}^2 = \alpha^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathcal{W}^3 = \alpha^3 \begin{bmatrix} Z_{ij} \\ 1 \\ 0 \end{bmatrix}.$$

Decompose $\Delta Q = (\Delta p, \Delta u, \Delta v)^T$:

$$\alpha_{i-1/2,j}^1 = (-\Delta Q^1 + Z\Delta Q^2)/(Z_{i-1,j} + Z_{ij}),$$

$$\alpha_{i-1/2,j}^2 = \Delta Q^3,$$

$$\alpha_{i-1/2,j}^3 = (\Delta Q^1 + Z_{i-1,j}\Delta Q^2)/(Z_{i-1,j} + Z_{ij}).$$

Fluctuations: (Note: $s^1 < 0$, $s^2 = 0$, $s^3 > 0$)

$$\mathcal{A}^- \Delta Q_{i-1/2,j} = s_{i-1/2,j}^1 \mathcal{W}_{i-1/2,j}^1,$$

$$\mathcal{A}^+ \Delta Q_{i-1/2,j} = s_{i-1/2,j}^3 \mathcal{W}_{i-1/2,j}^3.$$

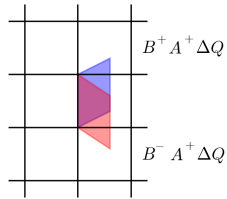
Notes:

Acoustics in heterogeneous media

Transverse solver: Split right-going fluctuation

$$\mathcal{A}^+ \Delta Q_{i-1/2,j} = s_{i-1/2,j}^3 \mathcal{W}_{i-1/2,j}^3$$

into up-going and down-going pieces:



Decompose $\mathcal{A}^+ \Delta Q_{i-1/2,j}$ into eigenvectors of B . **Down-going:**

$$\mathcal{A}^+ \Delta Q_{i-1/2,j} = \beta^1 \begin{bmatrix} -Z_{i,j-1} \\ 0 \\ 1 \end{bmatrix} + \beta^2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \beta^3 \begin{bmatrix} Z_{i,j} \\ 0 \\ 1 \end{bmatrix},$$

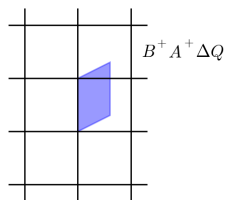
Notes:

Transverse solver for acoustics

Up-going part: $B^+ \mathcal{A}^+ \Delta Q_{i-1/2,j} = c_{i,j+1} \beta^3 r^3$ from

$$\mathcal{A}^+ \Delta Q_{i-1/2,j} = \beta^1 \begin{bmatrix} -Z_{ij} \\ 0 \\ 1 \end{bmatrix} + \beta^2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \beta^3 \begin{bmatrix} Z_{i,j+1} \\ 0 \\ 1 \end{bmatrix},$$

$$\beta^3 = ((\mathcal{A}^+ \Delta Q_{i-1/2,j})^1 + (\mathcal{A}^+ \Delta Q_{i-1/2,j})^3 Z_{i,j+1}) / (Z_{ij} + Z_{i,j+1}).$$



Notes:

Transverse Riemann solver in Clawpack

`rpt2` takes vector `asdq` and returns `bmasdq` and `bpasdq` where

$asdq = \mathcal{A}^* \Delta Q$ represents either

$\mathcal{A}^- \Delta Q$ if `imp` = 1, or

$\mathcal{A}^+ \Delta Q$ if `imp` = 2.

Returns $B^- \mathcal{A}^* \Delta Q$ and $B^+ \mathcal{A}^* \Delta Q$.

Note: there is also a parameter `ixy`:

`ixy` = 1 means normal solve was in x -direction,

`ixy` = 2 means normal solve was in y -direction,

In this case `asdq` represents $B^- \Delta Q$ or $B^+ \Delta Q$ and the routine must return $\mathcal{A}^- B^* \Delta Q$ and $\mathcal{A}^+ B^* \Delta Q$.

Notes:

Shallow water equations in two dimensions

$$\begin{aligned} h_t + (hu)_x + (hv)_y &= 0 \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y &= 0 \\ (hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y &= 0 \end{aligned}$$

Jacobian matrices:

$$f'(q) = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + gh & 2u & 0 \\ -uv & v & u \end{bmatrix}, \quad g'(q) = \begin{bmatrix} 0 & 0 & 1 \\ -uv & v & u \\ -v^2 + gh & 0 & 2v \end{bmatrix}.$$

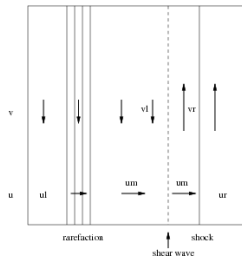
Eigenvalue and eigenvectors of $f'(q)$:

$$\begin{aligned} \lambda^{x1} &= u - c, & \lambda^{x2} &= u, & \lambda^{x3} &= u + c, \\ r^{x1} &= \begin{bmatrix} u \\ 1 \\ c \end{bmatrix}, & r^{x2} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, & r^{x3} &= \begin{bmatrix} u \\ 1 \\ -c \end{bmatrix}. \end{aligned}$$

Notes:

Structure of dam-break Riemann solution in 2d

$$h_l > h_r, \quad u_l = u_r = 0, \quad v_l < 0, \quad v_r > 0.$$



Jump in shear velocity v is advected with velocity u_m . (Linearly degenerate)

Note: Variations in v (y -velocity) in the x -direction do not compress fluid.

(Elasticity: restoring force from shear deformation \implies shear waves.)

Notes:

Shallow water equations in two dimensions

$$\begin{aligned} h_t + (hu)_x + (hv)_y &= 0 \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y &= 0 \\ (hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y &= 0 \end{aligned}$$

Roe averages:

$$\bar{h} = \frac{1}{2}(h_l + h_r), \quad \hat{u} = \frac{\sqrt{h_l}u_l + \sqrt{h_r}u_r}{\sqrt{h_l} + \sqrt{h_r}}, \quad \hat{v} = \text{similar.}$$

Roe matrix in x -direction:

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 \\ -\hat{u}^2 + g\bar{h} & 2\hat{u} & 0 \\ -\hat{u}\hat{v} & \hat{v} & \hat{u} \end{bmatrix} = f'(\hat{q})$$

Notes:

Shallow water equations in two dimensions

Roe matrix in x -direction:

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 \\ -\hat{u}^2 + g\bar{h} & 2\hat{u} & 0 \\ -\hat{u}\hat{v} & \hat{v} & \hat{u} \end{bmatrix},$$

has eigenvalues and eigenvectors

$$\begin{aligned} \hat{\lambda}^{x1} &= \hat{u} - \hat{c}, & \hat{\lambda}^{x2} &= \hat{u} & \hat{\lambda}^{x3} &= \hat{u} + \hat{c} \\ \hat{r}^{x1} &= \begin{bmatrix} 1 \\ \hat{u} - \hat{c} \\ \hat{v} \end{bmatrix}, & \hat{r}^{x2} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, & \hat{r}^{x3} &= \begin{bmatrix} 1 \\ \hat{u} + \hat{c} \\ \hat{v} \end{bmatrix} \end{aligned}$$

Transverse solver: use $\hat{v} \pm \hat{c}$ for transverse wave speeds.

http:

[//www.amath.washington.edu/~claw/applications/shallow/2d/rp/](http://www.amath.washington.edu/~claw/applications/shallow/2d/rp/)

Notes: