AMath 574 March 9, 2011

Today:

- Approximate Riemann solvers
- Multidimensional

Friday:

- AMR

Projects: Make an appointment this week, and see

`http://www.clawpack.org/links/burgersadv`

Shallow water equations

$h(x,t) =$ depth
$u(x,t) =$ velocity (depth averaged, varies only with x)

Conservation of mass and momentum hu gives system of two equations.

mass flux $= hu$,
momentum flux $= (hu)u + p$ where $p =$ hydrostatic pressure

$$h_t + (hu)_x = 0$$
$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$

Jacobian matrix:

$$f'(q) = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix}, \qquad \lambda = u \pm \sqrt{gh}.$$

Roe solver for Shallow Water

Given $h_l,\ u_l,\ h_r,\ u_r$, define

$$\bar{h} = \frac{h_l + h_r}{2}, \quad \hat{u} = \frac{\sqrt{h_l}u_l + \sqrt{h_r}u_r}{\sqrt{h_l} + \sqrt{hr}}$$

Then

$\hat{A}$ = Jacobian matrix evaluated at this average state

satisfies

$$A(q_r - q_l) = f(q_r) - f(q_l).$$

- Roe condition is satisfied,
- Isolated shock modeled well,
- Wave propagation algorithm is conservative,
- High resolution methods obtained using corrections with limited waves.

Roe solver for Shallow Water

Given $h_l,\ u_l,\ h_r,\ u_r$, define

$$\bar{h} = \frac{h_l + h_r}{2}, \quad \hat{u} = \frac{\sqrt{h_l}u_l + \sqrt{h_r}u_r}{\sqrt{h_l} + \sqrt{hr}}$$

Eigenvalues of $\hat{A} = f'(\hat{q})$ are:

$$\hat{\lambda}^1 = \hat{u} - \hat{c}, \quad \hat{\lambda}^2 = \hat{u} + \hat{c}, \quad \hat{c} = \sqrt{g\bar{h}}.$$

Eigenvectors:

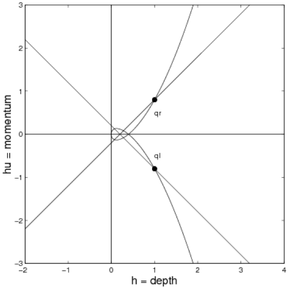
$$\hat{r}^1 = \begin{bmatrix} 1 \\ \hat{u} - \hat{c} \end{bmatrix}, \qquad \hat{r}^2 = \begin{bmatrix} 1 \\ \hat{u} + \hat{c} \end{bmatrix}.$$

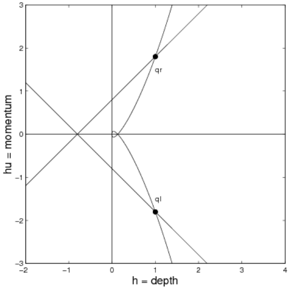
Examples in Clawpack 4.3 to be converted soon!

Potential failure of linearized solvers

Consider shallow water with $h_\ell = h_r$ and $u_r = -u_\ell \gg 1$.

Outflow away from interface $\implies$ small intermediate h_m.





With $u_r = 0.8$

Roe $h_m > 0$

With $u_r = 1.8$

Roe $h_m < 0$

HLL Solver

Harten – Lax – van Leer (1983): Use only 2 waves with

s^1 =minimum characteristic speed

s^2 =maximum characteristic speed

$$\mathcal{W}^1 = Q^* - Q_\ell, \qquad \mathcal{W}^2 = Q_r - Q^*$$

Conservation implies unique value for middle state Q^*:

$$s^1\mathcal{W}^1 + s^2\mathcal{W}^2 = f(Q_r) - f(Q_\ell)$$

$$\implies Q^* = \frac{f(Q_r) - f(Q_\ell) - s^2 Q_r + s^1 Q_\ell}{s^1 - s^2}.$$

HLL Solver

Harten – Lax – van Leer (1983): Use only 2 waves with
s^1 =minimum characteristic speed
s^2 =maximum characteristic speed

$$\mathcal{W}^1 = Q^* - Q_\ell, \qquad \mathcal{W}^2 = Q_r - Q^*$$

Conservation implies unique value for middle state Q^*:

$$s^1\mathcal{W}^1 + s^2\mathcal{W}^2 = f(Q_r) - f(Q_\ell)$$

$$\implies Q^* = \frac{f(Q_r) - f(Q_\ell) - s^2 Q_r + s^1 Q_\ell}{s^1 - s^2}.$$

Choice of speeds:

- Max and min of expected speeds over entire problem,
- Max and min of eigenvalues of $f'(Q_\ell)$ and $f'(Q_r)$.

HLLE Solver

Einfeldt: Choice of speeds for gas dynamics (or shallow water) that guarantees positivity.

Based on characteristic speeds and Roe averages:

$$s^1_{i-1/2} = \min_p(\min(\lambda^p_i, \hat{\lambda}^p_{i-1/2})),$$
$$s^2_{i-1/2} = \max_p(\max(\lambda^p_{i+1}, \hat{\lambda}^p_{i-1/2})).$$

where

λ^p_i is the pth eigenvalue of the Jacobian $f'(Q_i)$,

$\hat{\lambda}^p_{i-1/2}$ is the pth eigenvalue using Roe average $f'(\hat{Q}_{i-1/2})$

Wave propagation methods

- Solving Riemann problem gives waves $\mathcal{W}^p_{i-1/2}$,

$$Q_i - Q_{i-1} = \sum_p \mathcal{W}^p_{i-1/2}$$

  and speeds $s^p_{i-1/2}$. (Usually approximate solver used.)

- These waves update neighboring cell averages depending on sign of s^p (Godunov's method) via fluctuations.
- Waves also give characteristic decomposition of slopes:

$$q_x(x_{i-1/2}, t) \approx \frac{Q_i - Q_{i-1}}{\Delta x} = \frac{1}{\Delta x} \sum_p \mathcal{W}^p_{i-1/2}$$

- Apply limiter to each wave to obtain $\widetilde{\mathcal{W}}^p_{i-1/2}$.
- Use limited waves in second-order correction terms.

High-resolution wave-propagation algorithm

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(\mathcal{A}^-\Delta Q_{i+1/2} + \mathcal{A}^+\Delta Q_{i-1/2}) - \frac{\Delta t}{\Delta x}(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}),$$

where

$$\tilde{F}_{i-1/2} = \frac{1}{2}\sum_{p=1}^{m} |s_{i-1/2}^p| \left(1 - \frac{\Delta t}{\Delta x}|s_{i-1/2}^p|\right) \widetilde{\mathcal{W}}_{i-1/2}^p.$$

$\widetilde{\mathcal{W}}_{i-1/2}^p$ represents a limited version of the wave $\mathcal{W}_{i-1/2}^p$, obtained by comparing this jump with the jump $\mathcal{W}_{I-1/2}^p$ in the same family at the neighboring Riemann problem in the upwind direction,

$$I = \begin{cases} i-1 & \text{if } s_{i-1/2}^p > 0 \\ i+1 & \text{if } s_{i-1/2}^p < 0. \end{cases}$$

Wave limiters

Let $\mathcal{W}_{i-1/2} = Q_i^n - Q_{i-1}^n$.

Upwind: $Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}\mathcal{W}_{i-1/2}$.

Lax-Wendroff:

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}\mathcal{W}_{i-1/2} - \frac{\Delta t}{\Delta x}(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2})$$

$$\tilde{F}_{i-1/2} = \frac{1}{2}\left(1 - \left|\frac{u\Delta t}{\Delta x}\right|\right)|u|\mathcal{W}_{i-1/2}$$

High-resolution method:

$$\tilde{F}_{i-1/2} = \frac{1}{2}\left(1 - \left|\frac{u\Delta t}{\Delta x}\right|\right)|u|\widetilde{\mathcal{W}}_{i-1/2}$$

where $\widetilde{\mathcal{W}}_{i-1/2} = \Phi_{i-1/2}\mathcal{W}_{i-1/2}$.

Extension to linear systems

Approach 1: Diagonalize the system to

$$v_t + \Lambda v_x = 0$$

Apply scalar algorithm to each component.

Approach 2:

Solve the linear Riemann problem to decompose $Q_i^n - Q_{i-1}^n$ into waves.

Apply a wave limiter to each wave.

These are equivalent.

Important to apply limiters to waves or characteristic components, rather than to original variables.

Wave limiters for system

$Q_i - Q_{i-1}$ is split into waves $\mathcal{W}^p_{i-1/2} = \alpha^p_{i-1/2} r^p_{i-1/2} \in \mathbb{R}^m$.

Replace by $\widetilde{\mathcal{W}}^p_{i-1/2} = \Phi(\theta^p_{i-1/2})\mathcal{W}^p_{i-1/2}$ where

$$\theta^p_{i-1/2} = \frac{\mathcal{W}^p_{i-1/2} \cdot \mathcal{W}^p_{I-1/2}}{\mathcal{W}^p_{i-1/2} \cdot \mathcal{W}^p_{i-1/2}} \quad = \frac{\alpha^p_{I-1/2}}{\alpha^p_{i-1/2}} \quad \text{if } r^p_{i-1/2} = r^p_{I-1/2}$$

where

$$I = \begin{cases} i-1 & \text{if } s^p_{i-1/2} > 0 \\ i+1 & \text{if } s^p_{i-1/2} < 0. \end{cases}$$

In the scalar case this reduces to

$$\theta^1_{i-1/2} = \frac{\mathcal{W}^1_{I-1/2}}{\mathcal{W}^1_{i-1/2}} = \frac{Q_I - Q_{I-1}}{Q_i - Q_{i-1}}$$

First order hyperbolic PDE in 2 space dimensions

Advection equation: $q_t + uq_x + vq_y = 0$

First-order system: $q_t + Aq_x + Bq_y = 0$

where $q \in \mathbb{R}^m$ and $A, B \in \mathbb{R}^{m \times m}$.

Hyperbolic if $\cos(\theta)A + \sin(\theta)B$ is diagonalizable with real eigenvalues, for all angles θ.

First order hyperbolic PDE in 2 space dimensions

Advection equation: $q_t + uq_x + vq_y = 0$

First-order system: $q_t + Aq_x + Bq_y = 0$

where $q \in \mathbb{R}^m$ and $A, B \in \mathbb{R}^{m\times m}$.

Hyperbolic if $\cos(\theta)A + \sin(\theta)B$ is diagonalizable with real eigenvalues, for all angles θ.

This is required so that plane-wave data gives a 1d hyperbolic problem:

$$q(x, y, 0) = \breve{q}(x\cos\theta + y\sin\theta)$$ `(\breve q)`

implies contours of q in x–y plane are orthogonal to θ–direction.

Acoustics in 2 dimensions

$$\begin{aligned} p_t + K_0(u_x + v_y) &= 0 \\ \rho_0 u_t + p_x &= 0 \\ \rho_0 v_t + p_y &= 0 \end{aligned}$$

$$A = \begin{bmatrix} 0 & K_0 & 0 \\ 1/\rho_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad R^x = \begin{bmatrix} -Z_0 & 0 & Z_0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Solving $q_t + Aq_x = 0$ gives pressure waves in (p, u).
x-variations in v are stationary.

Acoustics in 2 dimensions

$$\begin{aligned} p_t + K_0(u_x + v_y) &= 0 \\ \rho_0 u_t + p_x &= 0 \\ \rho_0 v_t + p_y &= 0 \end{aligned}$$

$$A = \begin{bmatrix} 0 & K_0 & 0 \\ 1/\rho_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad R^x = \begin{bmatrix} -Z_0 & 0 & Z_0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Solving $q_t + Aq_x = 0$ gives pressure waves in (p, u).
x-variations in v are stationary.

$$B = \begin{bmatrix} 0 & 0 & K_0 \\ 0 & 0 & 0 \\ 1/\rho_0 & 0 & 0 \end{bmatrix} \qquad R^y = \begin{bmatrix} -Z_0 & 0 & Z_0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Solving $q_t + Bq_y = 0$ gives pressure waves in (p, v).
y-variations in u are stationary.

2d finite volume method

$$Q_{ij}^{n+1} = Q_{ij}^n - \frac{\Delta t}{\Delta x}[F_{i+1/2,j}^n - F_{i-1/2,j}^n] \\ - \frac{\Delta t}{\Delta y}[G_{i,j+1/2}^n - G_{i,j-1/2}^n].$$

Fluctuation form:

$$Q_{ij}^{n+1} = Q_{ij} - \frac{\Delta t}{\Delta x}(\mathcal{A}^+\Delta Q_{i-1/2,j} + \mathcal{A}^-\Delta Q_{i+1/2,j}) \\ - \frac{\Delta t}{\Delta y}(\mathcal{B}^+\Delta Q_{i,j-1/2} + \mathcal{B}^-\Delta Q_{i,j+1/2}) \\ - \frac{\Delta t}{\Delta x}(\tilde{F}_{i+1/2,j} - \tilde{F}_{i-1/2,j}) - \frac{\Delta t}{\Delta y}(\tilde{G}_{i,j+1/2} - \tilde{G}_{i,j-1/2}).$$

The $\tilde{F}$ and $\tilde{G}$ are correction fluxes to go beyond Godunov's upwind method.

Incorporate approximations to second derivative terms in each direction (q_{xx} and q_{yy}) and mixed term q_{xy}.

Wave propagation algorithms in 2D

Clawpack requires:

Normal Riemann solver `rpn2.f`
Solves 1d Riemann problem $q_t + Aq_x = 0$
Decomposes $\Delta Q = Q_{ij} - Q_{i-1,j}$ into $\mathcal{A}^+\Delta Q$ and $\mathcal{A}^-\Delta Q$.
For $q_t + Aq_x + Bq_y = 0$, split using eigenvalues, vectors:

$$A = R\Lambda R^{-1} \implies A^- = R\Lambda^- R^{-1}, A^+ = R\Lambda^+ R^{-1}$$

Input parameter `ixy` determines if it's in x or y direction.
In latter case splitting is done using B instead of A.
This is all that's required for dimensional splitting.

Wave propagation algorithms in 2D

Clawpack requires:

Normal Riemann solver `rpn2.f`
Solves 1d Riemann problem $q_t + Aq_x = 0$
Decomposes $\Delta Q = Q_{ij} - Q_{i-1,j}$ into $\mathcal{A}^+\Delta Q$ and $\mathcal{A}^-\Delta Q$.
For $q_t + Aq_x + Bq_y = 0$, split using eigenvalues, vectors:

$$A = R\Lambda R^{-1} \implies A^- = R\Lambda^- R^{-1}, A^+ = R\Lambda^+ R^{-1}$$

Input parameter `ixy` determines if it's in x or y direction.
In latter case splitting is done using B instead of A.
This is all that's required for dimensional splitting.

Transverse Riemann solver `rpt2.f`
Decomposes $\mathcal{A}^+\Delta Q$ into $\mathcal{B}^-\mathcal{A}^+\Delta Q$ and $\mathcal{B}^+\mathcal{A}^+\Delta Q$ by splitting this vector into eigenvectors of B.

(Or splits vector into eigenvectors of A if `ixy=2`.)

Acoustics in heterogeneous media

$$q_t + A(x,y)q_x + B(x,y)q_y = 0, \qquad q = (p,\ u,\ v)^T,$$

where

$$A = \begin{bmatrix} 0 & K(x,y) & 0 \\ 1/\rho(x,y) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & K(x,y) \\ 0 & 0 & 0 \\ 1/\rho(x,y) & 0 & 0 \end{bmatrix}.$$

Note: Not in conservation form!

Acoustics in heterogeneous media

$$q_t + A(x,y)q_x + B(x,y)q_y = 0, \qquad q = (p,\ u,\ v)^T,$$

where

$$A = \begin{bmatrix} 0 & K(x,y) & 0 \\ 1/\rho(x,y) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & K(x,y) \\ 0 & 0 & 0 \\ 1/\rho(x,y) & 0 & 0 \end{bmatrix}.$$

Note: Not in conservation form!

Wave propagation still makes sense. In x-direction:

$$\mathcal{W}^1 = \alpha^1 \begin{bmatrix} -Z_{i-1,j} \\ 1 \\ 0 \end{bmatrix}, \qquad \mathcal{W}^2 = \alpha^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathcal{W}^3 = \alpha^3 \begin{bmatrix} Z_{ij} \\ 1 \\ 0 \end{bmatrix}.$$

Wave speeds: $s^1_{i-1/2,j} = -c_{i-1,j},\ \ s^2_{i-1/2,j} = 0,\ \ s^3_{i-1/2,j} = +c_{ij}$.

Acoustics in heterogeneous media

$$\mathcal{W}^1 = \alpha^1 \begin{bmatrix} -Z_{i-1,j} \\ 1 \\ 0 \end{bmatrix}, \qquad \mathcal{W}^2 = \alpha^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathcal{W}^3 = \alpha^3 \begin{bmatrix} Z_{ij} \\ 1 \\ 0 \end{bmatrix}.$$

Decompose $\Delta Q = (\Delta p,\ \ \Delta u,\ \ \Delta v)^T$:

$$\begin{aligned}
\alpha^1_{i-1/2,j} &= (-\Delta Q^1 + Z\Delta Q^2)/(Z_{i-1,j} + Z_{ij}), \\
\alpha^2_{i-1/2,j} &= \Delta Q^3, \\
\alpha^3_{i-1/2,j} &= (\Delta Q^1 + Z_{i-1,j}\Delta Q^2)/(Z_{i-1,j} + Z_{ij}).
\end{aligned}$$

Acoustics in heterogeneous media

$$\mathcal{W}^1 = \alpha^1 \begin{bmatrix} -Z_{i-1,j} \\ 1 \\ 0 \end{bmatrix}, \qquad \mathcal{W}^2 = \alpha^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathcal{W}^3 = \alpha^3 \begin{bmatrix} Z_{ij} \\ 1 \\ 0 \end{bmatrix}.$$

Decompose $\Delta Q = (\Delta p,\ \Delta u,\ \Delta v)^T$:

$$\begin{aligned} \alpha^1_{i-1/2,j} &= (-\Delta Q^1 + Z \Delta Q^2)/(Z_{i-1,j} + Z_{ij}), \\ \alpha^2_{i-1/2,j} &= \Delta Q^3, \\ \alpha^3_{i-1/2,j} &= (\Delta Q^1 + Z_{i-1,j} \Delta Q^2)/(Z_{i-1,j} + Z_{ij}). \end{aligned}$$

Fluctuations: (Note: $s^1 < 0,\ s^2 = 0,\ s^3 > 0$)

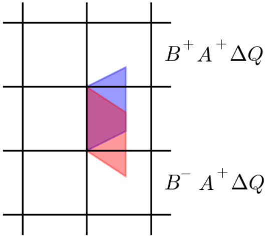
$$\mathcal{A}^- \Delta Q_{i-1/2,j} = s^1_{i-1/2,j} \mathcal{W}^1_{i-1/2,j},$$

$$\mathcal{A}^+ \Delta Q_{i-1/2,j} = s^3_{i-1/2,j} \mathcal{W}^3_{i-1/2,j}.$$

Transverse solver: Split right-going fluctuation

$$\mathcal{A}^+\Delta Q_{i-1/2,j} = s^3_{i-1/2,j}\mathcal{W}^3_{i-1/2,j}$$

into up-going and down-going pieces:



Decompose $\mathcal{A}^+\Delta Q_{i-1/2,j}$ into eigenvectors of B. Down-going:

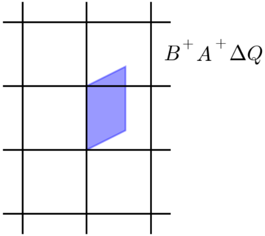
$$\mathcal{A}^+\Delta Q_{i-1/2,j} = \beta^1\begin{bmatrix}-Z_{i,j-1}\\0\\1\end{bmatrix} + \beta^2\begin{bmatrix}0\\-1\\0\end{bmatrix} + \beta^3\begin{bmatrix}Z_{ij}\\0\\1\end{bmatrix},$$

Transverse solver for acoustics

Up-going part: $\mathcal{B}^+\mathcal{A}^+\Delta Q_{i-1/2,j} = c_{i,j+1}\beta^3 r^3$ from

$$\mathcal{A}^+\Delta Q_{i-1/2,j} = \beta^1 \begin{bmatrix} -Z_{ij} \\ 0 \\ 1 \end{bmatrix} + \beta^2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \beta^3 \begin{bmatrix} Z_{i,j+1} \\ 0 \\ 1 \end{bmatrix},$$

$$\beta^3 = \left((\mathcal{A}^+\Delta Q_{i-1/2,j})^1 + (\mathcal{A}^+\Delta Q_{i-1/2,j})^3 Z_{i,j+1}\right) / (Z_{ij} + Z_{i,j+1}).$$



Transverse Riemann solver in Clawpack

`rpt2` takes vector `asdq` and returns `bmasdq` and `bpasdq` where

`asdq` $= \mathcal{A}^*\Delta Q$ represents either
  $\mathcal{A}^-\Delta Q$ if `imp = 1`, or
  $\mathcal{A}^+\Delta Q$ if `imp = 2`.

Returns $\mathcal{B}^-\mathcal{A}^*\Delta Q$ and $\mathcal{B}^+\mathcal{A}^*\Delta Q$.

Transverse Riemann solver in Clawpack

`rpt2` takes vector `asdq` and returns `bmasdq` and `bpasdq` where

`asdq` $= \mathcal{A}^*\Delta Q$ represents either
  $\mathcal{A}^-\Delta Q$ if `imp = 1`, or
  $\mathcal{A}^+\Delta Q$ if `imp = 2`.

Returns $\mathcal{B}^-\mathcal{A}^*\Delta Q$ and $\mathcal{B}^+\mathcal{A}^*\Delta Q$.

Note: there is also a parameter `ixy`:

- `ixy = 1` means normal solve was in x-direction,
- `ixy = 2` means normal solve was in y-direction,
  In this case `asdq` represents $\mathcal{B}^-\Delta Q$ or $\mathcal{B}^+\Delta Q$ and the routine must return $\mathcal{A}^-\mathcal{B}^*\Delta Q$ and $\mathcal{A}^+\mathcal{B}^*\Delta Q$.

Shallow water equations in two dimensions

$$
\begin{aligned}
h_t + (hu)_x + (hv)_y &= 0 \\
(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y &= 0 \\
(hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y &= 0
\end{aligned}
$$

Shallow water equations in two dimensions

$$\begin{aligned} h_t + (hu)_x + (hv)_y &= 0 \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y &= 0 \\ (hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y &= 0 \end{aligned}$$

Jacobian matrices:

$$f'(q) = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + gh & 2u & 0 \\ -uv & v & u \end{bmatrix}, \qquad g'(q) = \begin{bmatrix} 0 & 0 & 1 \\ -uv & v & u \\ -v^2 + gh & 0 & 2v \end{bmatrix}.$$

Shallow water equations in two dimensions

$$
\begin{aligned}
h_t + (hu)_x + (hv)_y &= 0 \\
(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y &= 0 \\
(hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y &= 0
\end{aligned}
$$

Jacobian matrices:

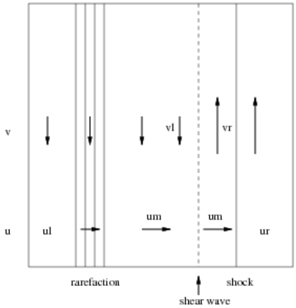
$$
f'(q) = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + gh & 2u & 0 \\ -uv & v & u \end{bmatrix}, \qquad g'(q) = \begin{bmatrix} 0 & 0 & 1 \\ -uv & v & u \\ -v^2 + gh & 0 & 2v \end{bmatrix}.
$$

Eigenvalue and eigenvectors of $f'(q)$:

$$
\begin{aligned}
\lambda^{x1} &= u - c, & \lambda^{x2} &= u, & \lambda^{x3} &= u + c, \\
r^{x1} &= \begin{bmatrix} 1 \\ u - c \\ v \end{bmatrix}, & r^{x2} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, & r^{x3} &= \begin{bmatrix} 1 \\ u + c \\ v \end{bmatrix}.
\end{aligned}
$$

Structure of dam-break Riemann solution in 2d

$$h_l > h_r, \qquad u_l = u_r = 0, \qquad v_l < 0,\ v_r > 0.$$

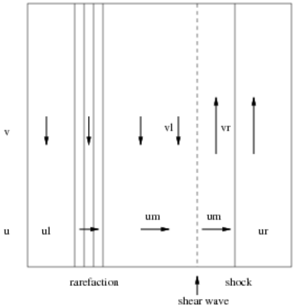


Jump in shear velocity v is advected with velocity u_m. (Linearly degenerate)

Note: Variations in v (y-velocity) in the x-direction do not compress fluid.

Structure of dam-break Riemann solution in 2d

$$h_l > h_r, \qquad u_l = u_r = 0, \qquad v_l < 0, \; v_r > 0.$$



Jump in shear velocity v is advected with velocity u_m. (Linearly degenerate)

Note: Variations in v (y-velocity) in the x-direction do not compress fluid.

(Elasticity: restoring force from shear deformation $\implies$ shear waves.)

Shallow water equations in two dimensions

$$
\begin{aligned}
h_t + (hu)_x + (hv)_y &= 0 \\
(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y &= 0 \\
(hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y &= 0
\end{aligned}
$$

Shallow water equations in two dimensions

$$\begin{aligned} h_t + (hu)_x + (hv)_y &= 0 \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y &= 0 \\ (hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y &= 0 \end{aligned}$$

Roe averages:

$$\bar{h} = \frac{1}{2}(h_l + h_r), \qquad \hat{u} = \frac{\sqrt{h_l}\,u_l + \sqrt{h_r}\,u_r}{\sqrt{h_l} + \sqrt{h_r}}, \hat{v} = \text{similar.}$$

Roe matrix in x-direction:

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 \\ -\hat{u}^2 + g\bar{h} & 2\hat{u} & 0 \\ -\hat{u}\hat{v} & \hat{v} & \hat{u} \end{bmatrix} = f'(\hat{q})$$

Shallow water equations in two dimensions

Roe matrix in x-direction:

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 \\ -\hat{u}^2 + g\bar{h} & 2\hat{u} & 0 \\ -\hat{u}\hat{v} & \hat{v} & \hat{u} \end{bmatrix},$$

has eigenvalues and eigenvectors

$$\begin{aligned} \hat{\lambda}^{x1} &= \hat{u} - \hat{c}, & \hat{\lambda}^{x2} &= \hat{u} & \hat{\lambda}^{x3} &= \hat{u} + \hat{c} \\ \hat{r}^{x1} &= \begin{bmatrix} 1 \\ \hat{u} - \hat{c} \\ \hat{v} \end{bmatrix}, & \hat{r}^{x2} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, & \hat{r}^{x3} &= \begin{bmatrix} 1 \\ \hat{u} + \hat{c} \\ \hat{v} \end{bmatrix} \end{aligned}$$

Shallow water equations in two dimensions

Roe matrix in x-direction:

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 \\ -\hat{u}^2 + g\bar{h} & 2\hat{u} & 0 \\ -\hat{u}\hat{v} & \hat{v} & \hat{u} \end{bmatrix},$$

has eigenvalues and eigenvectors

$$\begin{aligned} \hat{\lambda}^{x1} &= \hat{u} - \hat{c}, & \hat{\lambda}^{x2} &= \hat{u} & \hat{\lambda}^{x3} &= \hat{u} + \hat{c} \\ \hat{r}^{x1} &= \begin{bmatrix} 1 \\ \hat{u} - \hat{c} \\ \hat{v} \end{bmatrix}, & \hat{r}^{x2} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, & \hat{r}^{x3} &= \begin{bmatrix} 1 \\ \hat{u} + \hat{c} \\ \hat{v} \end{bmatrix} \end{aligned}$$

Transverse solver: use $\hat{v} \pm \hat{c}$ for transverse wave speeds.

http://www.amath.washington.edu/~claw/applications/shallow/2d/rp/