Conservation Laws and Finite Volume Methods

AMath 574
Winter Quarter, 2011

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Outline

Today:

- 2D hyperbolic systems (on board)
- 2D Advection (on board)
- Clawpack
- Acoustics
- 2D examples

Friday:

- Software: Clawpack prerequisites and VM

Reading: Start Chapter 3 for Monday

CLAWPACK — Conservation Laws Package

- Open source, 1d, 2d, 3d **www.clawpack.org**
- Originally f77 with Matlab graphics.
- Moving to f95 with Python.
- Adaptive mesh refinement.
- OpenMP and MPI.

User supplies:

- Riemann solver, splitting data into waves and speeds
  (Need not be in conservation form)
- Boundary condition routine to extend data to ghost cells
  Standard `bc1.f` routine includes many standard BC's
- Initial conditions — `qinit.f`
- Source terms — `src1.f`

Some applications where CLAWPACK has been used

- Aerodynamics, supersonic flows
- Seismic waves, tsunamis, flow on the sphere
- Volcanic flows, dusty gas jets, pyroclastic surges
- Ultrasound, lithotripsy, shock wave therapy
- Plasticity, nonlinear elasticity
- Chemotaxis and pattern formation
- Semiconductor modeling
- Multi-fluids, multi-phase flows, bubbly flow
- Combustion, detonation waves
- Astrophysics: binary stars, planetary nebulae, jets,
- Magnetohydrodynamics, plasmas, relativistic flow
- Numerical relativity — gravitational waves, cosmology

Options for using Clawpack

1. Install from tar file or Subversion: **Instructions**.

   Requires some **prerequisites:** Fortran, Python modules.

   rerequisites exist on some AMath Linux computers.

2. Use the **VirtualClaw** virtual machine.
3. For some applications, use **EagleClaw**
   (Easy Access Graphical Laboratory for Exploring Conservation Laws)

Read the **documentation**!

Also perhaps useful:

AMath 583 Class notes on Python, Fortran, version control, etc.

Advection examples

- **$CLAW/apps/advection/1d/example1/README.html**
- **Advection in EagleClaw**

Example: Linear acoustics in a 1d tube

$$q = \begin{bmatrix} p \\ u \end{bmatrix} \qquad \begin{array}{l} p(x,t) = \text{pressure perturbation} \\ u(x,t) = \text{velocity} \end{array}$$

Equations:

$$\begin{aligned} p_t + \kappa u_x &= 0 \qquad & \kappa &= \text{bulk modulus} \\ \rho u_t + p_x &= 0 \qquad & \rho &= \text{density} \end{aligned}$$

or

$$\begin{bmatrix} p \\ u \end{bmatrix}_t + \begin{bmatrix} 0 & \kappa \\ 1/\rho & 0 \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_x = 0.$$

Eigenvalues: $\lambda = \pm c$, where $c = \sqrt{\kappa/\rho} =$ sound speed

Second order form: Can combine equations to obtain

$$p_{tt} = c^2 p_{xx}$$

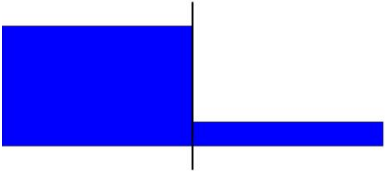
Riemann Problem

Special initial data:

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$

Example: Acoustics with bursting diaphram

Pressure:



Acoustic waves propagate with speeds $\pm c$.

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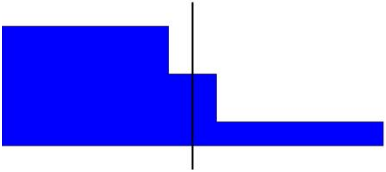

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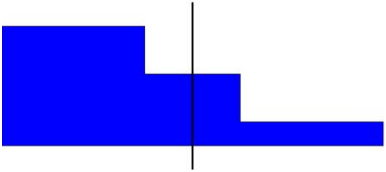

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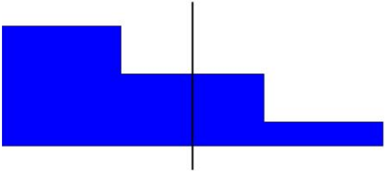

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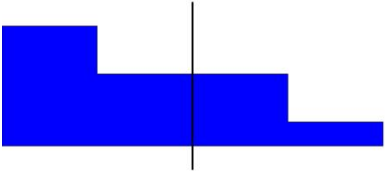

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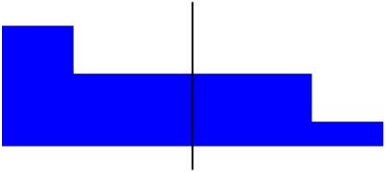

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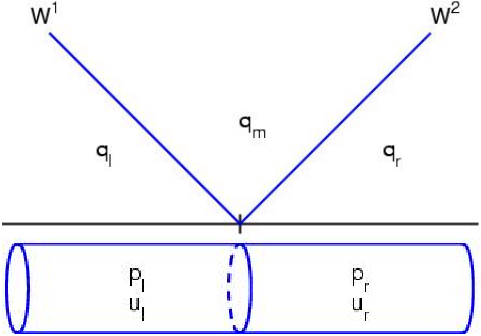
Riemann Problem for acoustics

Waves propagating in x–t space:



Left-going wave $\mathcal{W}^1 = q_m - q_l$ and
right-going wave $\mathcal{W}^2 = q_r - q_m$ are eigenvectors of A.

Acoustics examples

- **$CLAW/apps/acoustics/1d/example2/README.html**
- **Acoustics in EagleClaw**

Two-dimensional advection examples

See the Clawpack gallery:
www.clawpack.org/doc/apps.html

Equations of linear elasticity (in 2d)

$$
\begin{aligned}
\sigma^{11}_t &\quad - (\lambda + 2\mu) u_x - \lambda v_y &= 0 \\
\sigma^{22}_t &\quad - \lambda u_x - (\lambda + 2\mu) v_y &= 0 \\
\sigma^{12}_t &\quad - \mu(v_x + u_y) &= 0 \\
\rho u_t &\quad - \sigma^{11}_x - \sigma^{12}_y &= 0 \\
\rho v_t &\quad - \sigma^{12}_x - \sigma^{22}_y &= 0
\end{aligned}
$$

where $\lambda(x, y)$ and $\mu(x, y)$ are Lamé parameters.

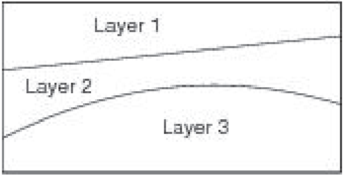
This has the form $q_t + Aq_x + Bq_y = 0$.

The matrix $(A\cos\theta + B\sin\theta)$ has eigenvalues $\;-c_p,\;\; -c_s,\;\; 0,\;\; c_s,\;\; c_p$

P-wave (dilatational) speed: $\quad c_p = \sqrt{\frac{\lambda+2\mu}{\rho}}$

S-wave (shear) speed: $\quad c_s = \sqrt{\frac{\mu}{\rho}}$

Seismic waves in layered earth



Layers 1 and 3: $\rho = 2, \;\; \lambda = 1, \;\; \mu = 1, \;\; c_p \approx 1.2, \;\; c_s \approx 0.7$

Layer 2: $\rho = 5, \;\; \lambda = 10, \;\; \mu = 5, \;\; c_p = 2.0, \;\; c_s = 1$

Impulse at top surface at $t = 0$.

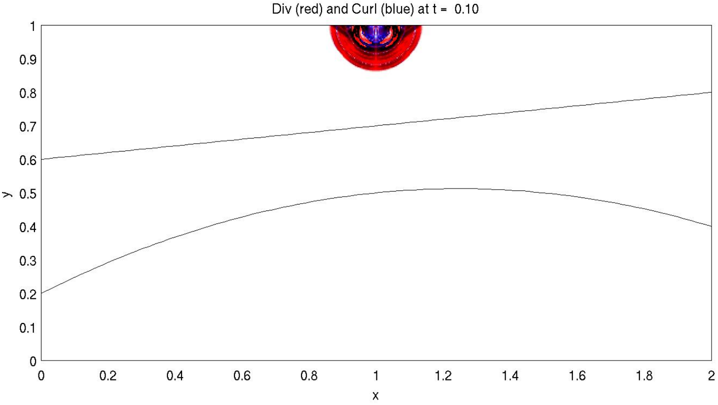
Solved on uniform Cartesian grid (600×300).

Cell average of material parameters used in each finite volume cell.

Extrapolation at computational boundaries.

Seismic wave in layered medium

Red = div(u) [P-waves],   Blue = curl(u) [S-waves]



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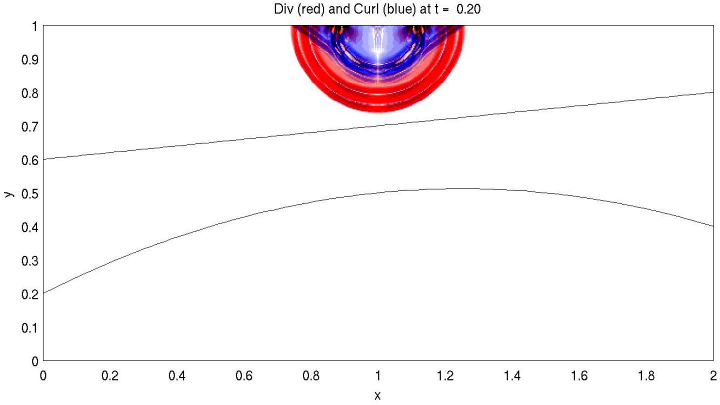

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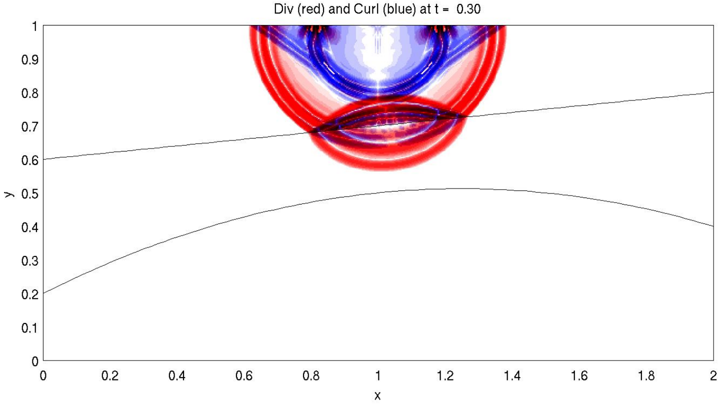

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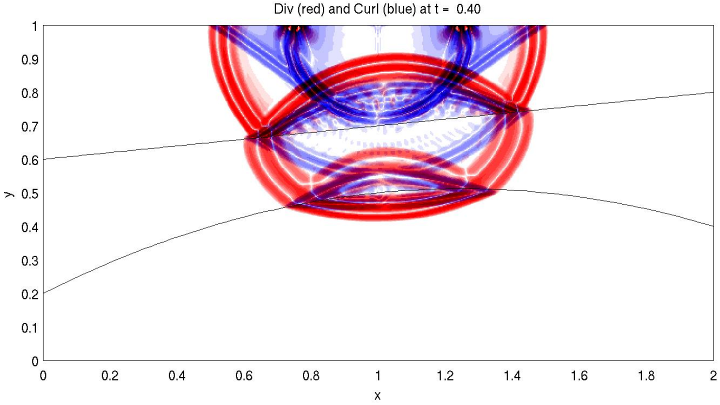

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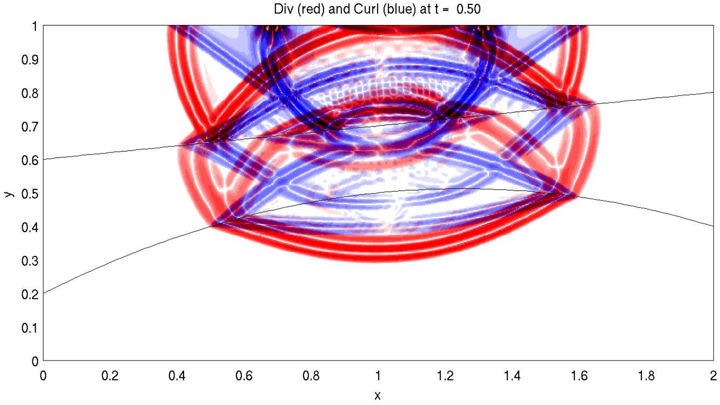

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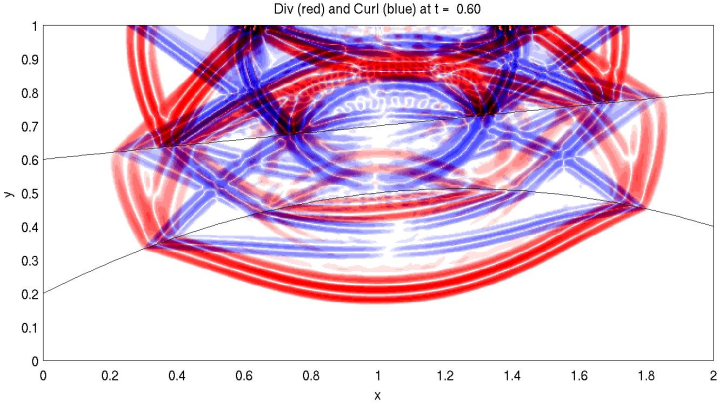

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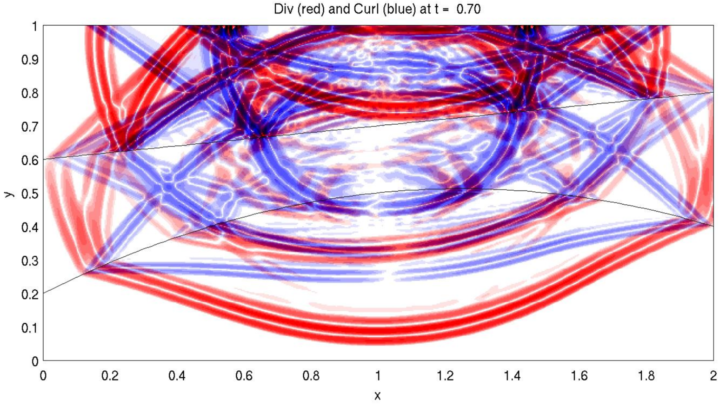

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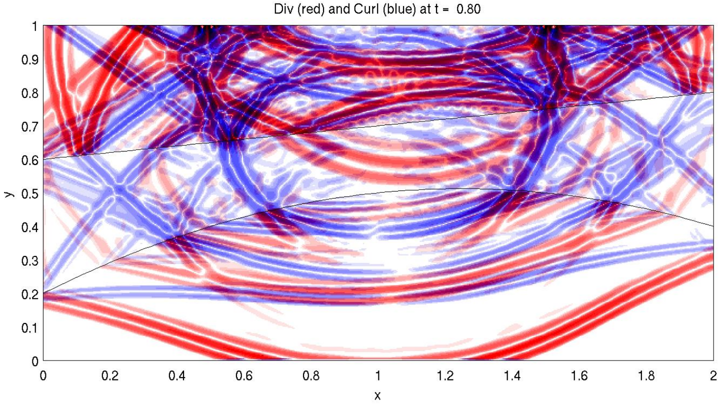

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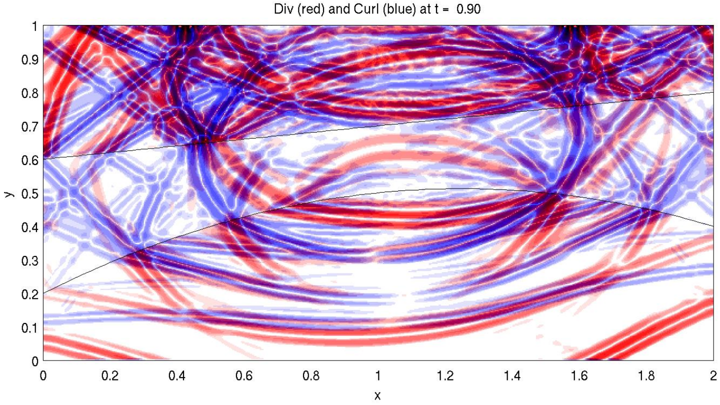

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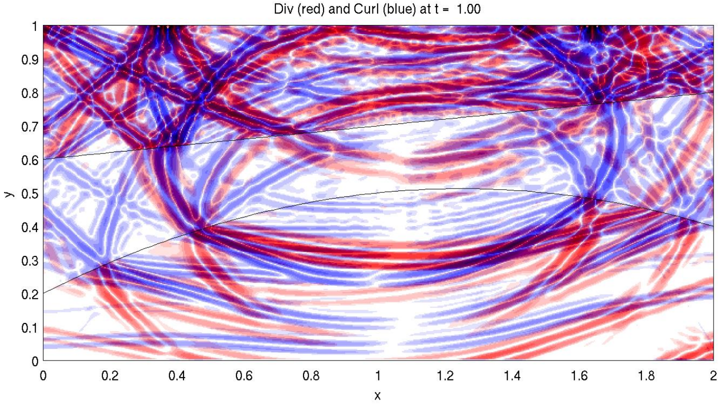
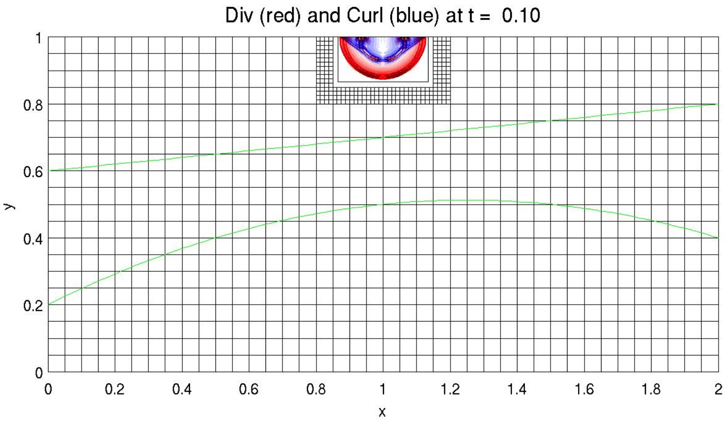
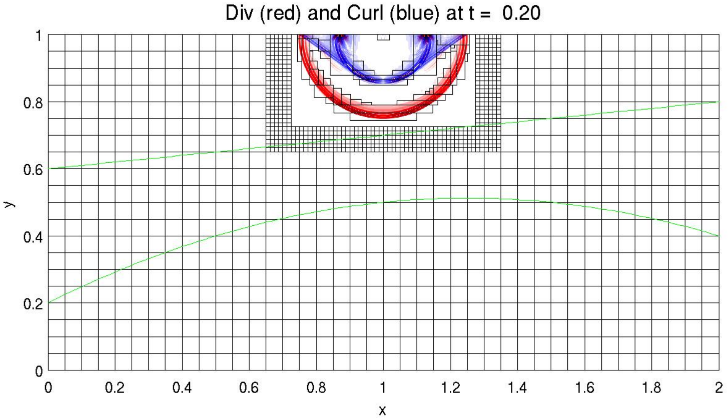
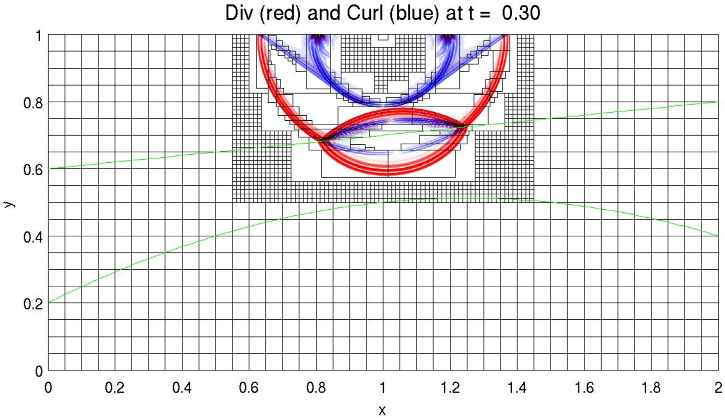
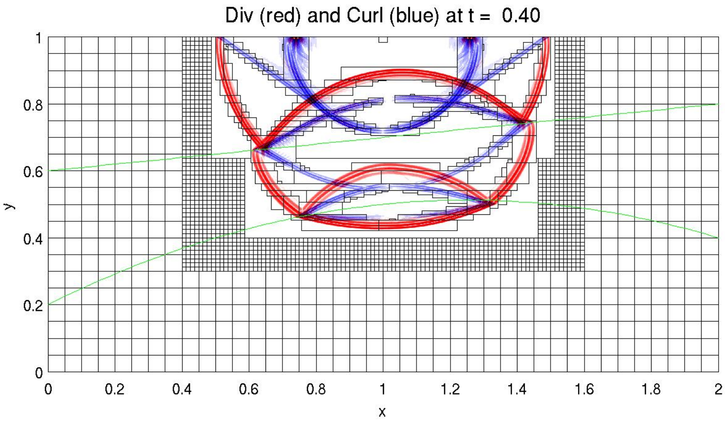



Div (red) and Curl (blue) at t = 0.10
1
0.8
0.6
0.4
0.2
0
y
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
x


Div (red) and Curl (blue) at t = 0.20
y
x
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2



Div (red) and Curl (blue) at t = 0.30
1
0.8
0.6
0.4
0.2
0
y
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
x


Div (red) and Curl (blue) at t = 0.40
y
x
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2