Conservation Laws and Finite Volume Methods

AMath 574
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Course outline

Main goals:

- Theory of hyperbolic conservation laws in one dimension
- Finite volume methods in 1 and 2 dimensions
- Some applications: advection, acoustics, Burgers', shallow water equations, gas dynamics, traffic flow
- Use of the Clawpack software: **www.clawpack.org**

Slides will be posted and green links can be clicked.

`http://kingkong.amath.washington.edu/trac/am574w11`

Outline

Today:

- Hyperbolic equations
- Advection
- Riemann problem
- Diffusion
- Clawpack
- Acoustics

Reading: Chapters 1 and 2

First order hyperbolic PDE in 1 space dimension

Linear: $q_t + Aq_x = 0, \qquad q(x,t) \in \mathbb{R}^m,\ A \in \mathbb{R}^{m\times m}$

Conservation law: $q_t + f(q)_x = 0, \qquad f : \mathbb{R}^m \to \mathbb{R}^m$ (flux)

Quasilinear form: $q_t + f'(q)q_x = 0$

Hyperbolic if A or $f'(q)$ is diagonalizable with real eigenvalues.

Models wave motion or advective transport.

Eigenvalues are wave speeds.

Note: Second order wave equation $p_{tt} = c^2 p_{xx}$ can be written as a first-order system (acoustics).

Derivation of Conservation Laws

$q(x,t) =$ density function for some conserved quantity, so

$$\int_{x_1}^{x_2} q(x,t)\,dx = \text{total mass in interval}$$

changes only because of fluxes at left or right of interval.

Derivation of Conservation Laws

$q(x, t) =$ density function for some conserved quantity.

Integral form:

$$\frac{d}{dt} \int_{x_1}^{x_2} q(x, t)\, dx = F_1(t) - F_2(t)$$

where

$$F_j = f(q(x_j, t)), \qquad f(q) = \text{flux function.}$$

Derivation of Conservation Laws

If q is smooth enough, we can rewrite

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = f(q(x_1,t)) - f(q(x_2,t))$$

as

$$\int_{x_1}^{x_2} q_t\,dx = -\int_{x_1}^{x_2} f(q)_x\,dx$$

or

$$\int_{x_1}^{x_2} (q_t + f(q)_x)\,dx = 0$$

True for all $x_1,\ x_2 \implies$ differential form:

$$q_t + f(q)_x = 0.$$

Finite differences vs. finite volumes

Finite difference Methods

- Pointwise values $Q_i^n \approx q(x_i, t_n)$
- Approximate derivatives by finite differences
- Assumes smoothness

Finite volume Methods

- Approximate cell averages: $Q_i^n \approx \dfrac{1}{\Delta x} \displaystyle\int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t_n)\, dx$
- Integral form of conservation law,

$$\frac{\partial}{\partial t} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)\, dx \quad = \quad f(q(x_{i-1/2}, t)) - f(q(x_{i+1/2}, t))$$

leads to conservation law $q_t + f_x = 0$ but also directly to numerical method.

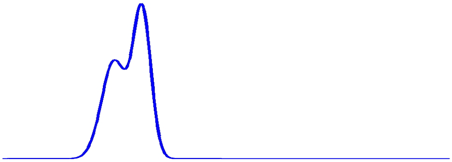
Advection equation

$u =$ constant flow velocity

$q(x,t) =$ tracer concentration, $\quad f(q) = uq$

$\implies \quad q_t + uq_x = 0.$

True solution: $q(x,t) = q(x - ut, 0)$



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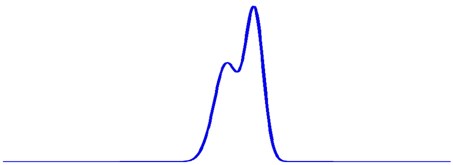

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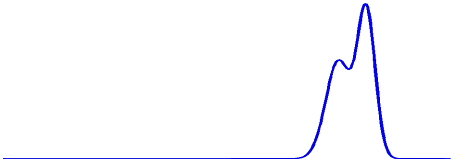


Characteristics for advection

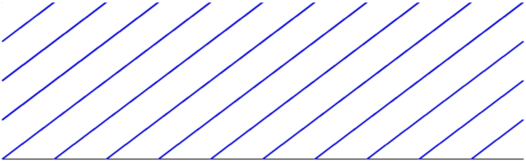
$q(x,t) = \eta(x-ut) \implies q$ is constant along lines

$$X(t) = x_0 + ut, \quad t \geq 0.$$

Can also see that q is constant along $X(t)$ from:

$$\begin{aligned}\frac{d}{dt}q(X(t),t) &= q_x(X(t),t)X'(t) + q_t(X(t),t)\\ &= q_x(X(t),t)u + q_t(X(t),t)\\ &= 0.\end{aligned}$$

In x–t plane:



Cauchy problem for advection

Advection equation on infinite 1D domain:

$$q_t + uq_x = 0 \qquad -\infty < x < \infty, \;\; t \geq 0,$$

with initial data

$$q(x,0) = \eta(x) \qquad -\infty < x < \infty.$$

Solution:

$$q(x,t) = \eta(x - ut) \qquad -\infty < x < \infty, \;\; t \geq 0.$$

Initial–boundary value problem (IBVP) for advection

Advection equation on finite 1D domain:

$$q_t + uq_x = 0 \qquad a < x < b, \quad t \geq 0,$$

with initial data

$$q(x,0) = \eta(x) \qquad a < x < b.$$

and boundary data at the inflow boundary:

If $u > 0$, need data at $x = a$:

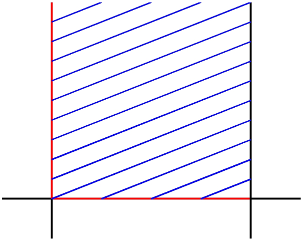
$$q(a,t) = g(t), \qquad t \geq 0,$$

If $u < 0$, need data at $x = b$:

$$q(b,t) = g(t), \qquad t \geq 0,$$

Characteristics for IBVP

In x–t plane for the case $u > 0$:



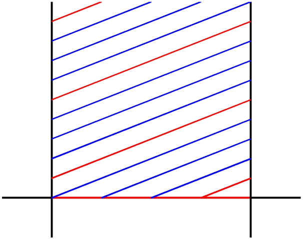
Solution:

$$q(x,t) = \begin{cases} \eta(x - ut) & \text{if } a \le x - ut \le b, \\ g((x-a)/u) & \text{otherwise} \end{cases}.$$

Periodic boundary conditions

$$q(a,t) = q(b,t), \qquad t \geq 0.$$

In x–t plane for the case $u > 0$:



Solution:

$$q(x,t) = \eta(X_0(x,t)),$$

where $X_0(x,t) = a + \text{mod}(x - ut - a,\ b - a)$.

The Riemann problem

The Riemann problem consists of the hyperbolic equation under study together with initial data of the form

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x \geq 0 \end{cases}$$

Piecewise constant with a single jump discontinuity from q_l to q_r.

The Riemann problem is fundamental to understanding

- The mathematical theory of hyperbolic problems,
- Godunov-type finite volume methods

Why? Even for nonlinear systems of conservation laws, the Riemann problem can often be solved for general q_l and q_r, and consists of a set of waves propagating at constant speeds.

The Riemann problem for advection

The Riemann problem for the advection equation $q_t + uq_x = 0$ with

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x \ge 0 \end{cases}$$

has solution

$$q(x,t) = q(x-ut,0) = \begin{cases} q_l & \text{if } x < ut \\ q_r & \text{if } x \ge ut \end{cases}$$

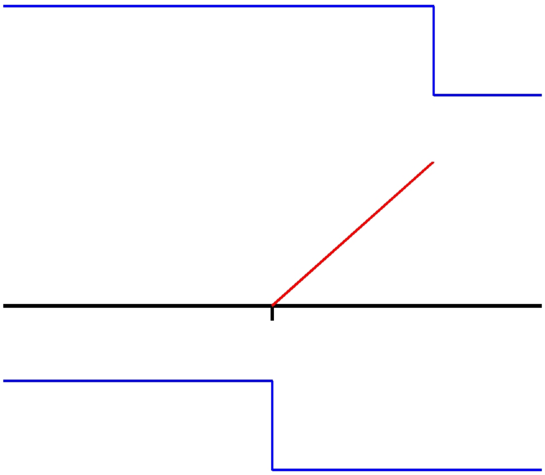
consisting of a single wave of strength $\mathcal{W}^1 = q_r - q_l$ propagating with speed $s^1 = u$.

Riemann solution for advection

$q(x, T)$

x–t plane

$q(x, 0)$



Discontinuous solutions

Note: The Riemann solution is not a classical solution of the PDE $q_t + uq_x = 0$, since q_t and q_x blow up at the discontinuity.

Integral form:

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = uq(x_1,t) - uq(x_2,t)$$

Integrate in time from t_1 to t_2 to obtain

$$\int_{x_1}^{x_2} q(x,t_2)\,dx - \int_{x_1}^{x_2} q(x,t_1)\,dx$$
$$= \int_{t_1}^{t_2} uq(x_1,t)\,dt - \int_{t_1}^{t_2} uq(x_2,t)\,dt.$$

The Riemann solution satisfies the given initial conditions and this integral form for all $x_2 > x_1$ and $t_2 > t_1 \geq 0$.

Diffusive flux

$q(x,t) =$ concentration
$\beta =$ diffusion coefficient ($\beta > 0$)

diffusive flux $= -\beta q_x(x,t)$

$q_t + f_x = 0 \implies$ diffusion equation:

$$q_t = (\beta q_x)_x = \beta q_{xx} \text{ (if } \beta = \text{const).}$$

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Heat equation: Same form, where

$q(x,t) =$ density of thermal energy $= \kappa T(x,t)$,
$T(x,t) =$ temperature, $\kappa =$ heat capacity,
flux $= -\beta T(x,t) = -(\beta/\kappa) q(x,t) \implies$

$$q_t(x,t) = (\beta/\kappa) q_{xx}(x,t).$$

Advection-diffusion

$q(x,t) =$ concentration that advects with velocity u
and diffuses with coefficient β:

$$\text{flux} = uq - \beta q_x.$$

Advection-diffusion equation:

$$q_t + uq_x = \beta q_{xx}.$$

If $\beta > 0$ then this is a parabolic equation.

Advection dominated if u/β (the Péclet number) is large.

Fluid dynamics: "parabolic terms" arise from

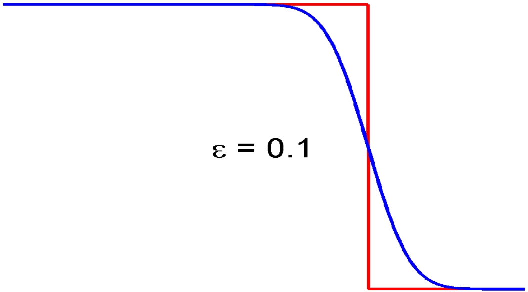
- thermal diffusion and
- diffusion of momentum, where the diffusion parameter is the viscosity.

Discontinuous solutions

Vanishing Viscosity solution: The Riemann solution $q(x,t)$ is the limit as $\epsilon \to 0$ of the solution $q^\epsilon(x,t)$ of the parabolic advection-diffusion equation

$$q_t + uq_x = \epsilon q_{xx}.$$

For any $\epsilon > 0$ this has a classical smooth solution:



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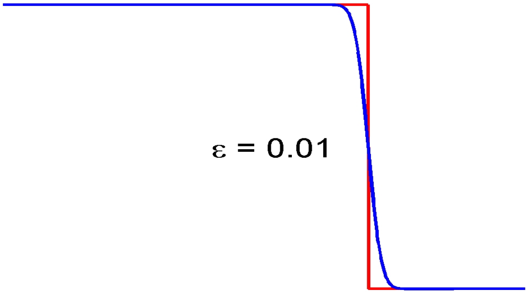

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