AMath 584 / Math 584
Homework #4
Due 11:00pm PDT
Tuesday, November 29, 2016

Name: Your Name Here

Netid: Your NetID Here

Problem 1.

Exercise 20.2 in the book. Note that assuming the conditions of Exercise 20.1 just means that it is valid to assume you can do Gaussian elimination without pivoting, which is assumed for this problem. (Pivoting would change the sparsity pattern of the result.)

Problem 2.

(a) Consider the matrix

$$A = \begin{bmatrix} 4 & -1 & -1 & -1 \\ 1 & -4 & 1 & 1 \\ 1 & 1 & -4 & 1 \\ 1 & 1.5 & 1.5 & -4 \end{bmatrix}$$

Apply one step of Gaussian elimination to introduce zeros in the first column. You do not need to pivot since $|a_{11}|$ is the maximum value in the first column. What matrix A_1 do you obtain?

(b) Let $\tilde{A}_1$ be the 3×3 submatrix in the lower right corner of A_1. This is the submatrix we work with in the next step of Gaussian elimination. Note that again you do not need to pivot.

Verify that both the original matrix A and the matrix $\tilde{A}_1$ are *strictly column diagonally dominant* in the sense defined in Exercise 21.6 in the book (abbreviated SCDD below).

(c) Now do Exercise 21.6. The above example is intended to help you think about how to approach this. Show that in general if $A \in \mathbb{C}^{m\times m}$ is SCDD then so is the $(m-1)\times(m-1)$ submatrix obtained after one step of Gaussian elimination. You can then apply induction.

Note that when you apply elimination, the diagonal terms might decrease in magnitude while the off-diagonal terms might increase in magnitude. But since we are only looking at a submatrix there is one fewer off-diagonal, and you can bound these decreases and increases (using the SCCD property of the original matrix) to obtain the bounds required to show that $\tilde{A}_1$ is SCCD. You might want to play around with the example above to understand why this should work.

Problem 3.

(a) Suppose $A \in \mathbb{C}^{m\times m}$ is nonsingular. Show that $\|A^*A\|_2 = \|A\|_2^2$ and similarly for the inverse of this matrix. Conclude that $\kappa_2(A^*A) = \kappa_2(A)^2$.

(b) Show that A^*A is always hermitian positive definite, provided A is nonsingular. Is the same true if $\text{rank}(A) < m$? Explain why or why not.

Problem 4.

Suppose $A \in \mathbb{C}^{m\times m}$ is hermitian positive definite and $A = R^*R$ is its Cholesky factorization. Show that $\|R\|_2 = \|R^*\|_2 = \|A\|_2^{1/2}$ by using the SVD of R.

Problem 5.

Exercise 23.2.

Hints:

Note that the proof of Theorem 16.2 is suggested as a guide since it also involves combining different backward error analyses into a new result. In this exercise the point is to show that if we compute $\tilde{R}$ by a Cholesky decomposition (numerically) and then compute $\tilde{y}$ by solving $\tilde{R}^*y = b$ numerically and finally compute $\tilde{x}$ by solving $\tilde{R}x = \tilde{y}$ numerically, then we can get the bound (23.6) showing that the final computed $\tilde{x}$ is the exact solution to a nearby problem.

Theorem 17.1 says that solving an upper triangular system by back substitution is backward stable and gives a result you will need to use. The same result holds for solving a lower triangular system by forward substitution, and you need to do both for this problem.

This means you can assume for example that $\tilde{y}$ satisfies exactly an equation of the form

$$(\tilde{R}^* + \delta\tilde{R}^*)\tilde{y} = b$$

and then $\tilde{x}$ satisfies an equation of the form

$$(\tilde{R} + \delta\hat{R})\tilde{x} = \tilde{y}$$

where $\delta\tilde{R}$ and $\delta\hat{R}$ are two different perturbations but both satisfying similar bounds. Now combine things (using also Theorem 23.2 at this point) to obtain the desired bound, keeping in mind the result of Problem 4.