Conservation Laws and Finite Volume Methods

AMath 574
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http://faculty.washington.edu/rjl/classes/am574w2017

Outline

- Riemann solution for shallow water:
  http://faculty.washington.edu/rjl/classes/am574w2017/riemann.html
- Euler equations of gas dynamics
- Notebooks from **https://github.com/clawpack/riemann_book**

Reading: Chapter 14

Compressible gas dynamics

In one space dimension (e.g. in a pipe).
$\rho(x,t) =$ density, $\quad u(x,t) =$ velocity,
$p(x,t) =$ pressure, $\quad \rho(x,t)u(x,t) =$ momentum.

Conservation of:

mass:	ρ	flux:	ρu
momentum:	ρu	flux:	$(\rho u)u + p$
(energy)			

Conservation laws:

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

Equation of state:

$$p = P(\rho).$$

(Later: p may also depend on internal energy / temperature)

Compressible gas dynamics

Conservation laws:

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

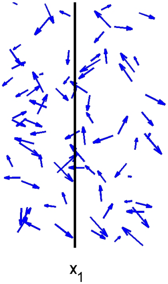
Momentum flux:

$\rho u^2 = (\rho u)u =$ advective flux

p term in flux?

- $-p_x =$ force in Newton's second law,
- as momentum flux: microscopic motion of gas molecules.

Momentum flux arising from pressure



Note that:

- molecules with positive x-velocity crossing x_1 to right increase the momentum in $[x_1, x_2]$
- molecules with negative x-velocity crossing x_1 to left also increase the momentum in $[x_1, x_2]$

Hence momentum flux increases with pressure $p(x_1, t)$ even if macroscopic (average) velocity is zero.

Compressible gas dynamics

Conservation laws:

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Equation of state:

$$p = P(\rho).$$

Same as shallow water if $P(\rho) = \frac{1}{2} g \rho^2$ (with $\rho \equiv h$).

Isothermal: $P(\rho) = a^2 \rho$ (since T proportional to p/ρ).

Isentropic: $P(\rho) = \hat{\kappa} \rho^\gamma$ ($\gamma \approx 1.4$ for air)

Jacobian matrix:

$$f'(q) = \begin{bmatrix} 0 & 1 \\ P'(\rho) - u^2 & 2u \end{bmatrix}, \qquad \lambda = u \pm \sqrt{P'(\rho)}.$$

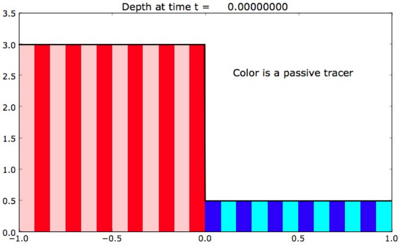
The Riemann problem

Dam break problem for shallow water equations with tracer

$$h_t + (hu)_x = 0$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$

$$(h\phi)_t + (uh\phi)_x = 0$$



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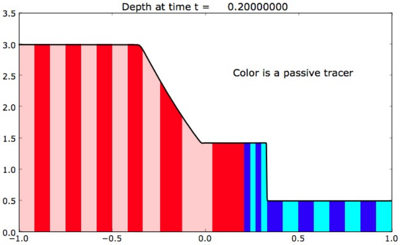

The Riemann problem

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$$
\begin{aligned}
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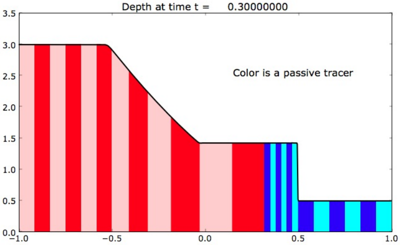

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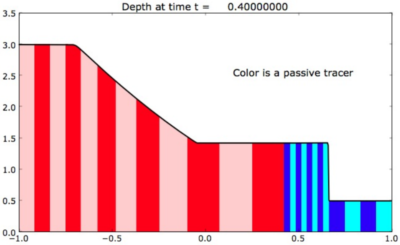

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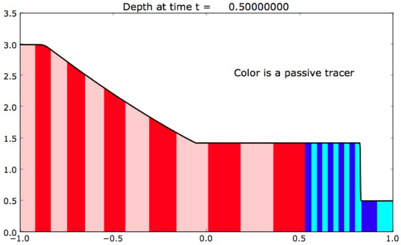
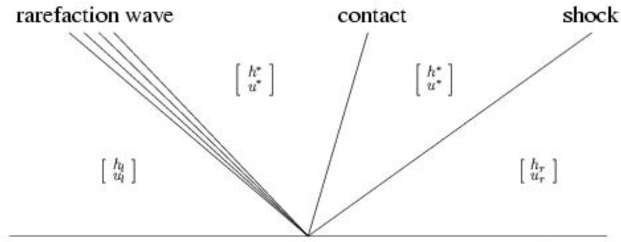


Riemann solution for the SW equations in x-t plane



Similarity solution:

Solution is constant on any ray: $q(x,t) = Q(x/t)$

Riemann solution can be calculated for many problems.
Linear: Eigenvector decomposition. Nonlinear: more difficult.

In practice "approximate Riemann solvers" used numerically.

Euler equations of gas dynamics

Conservation of mass, momentum, energy: $q_t + f(q)_x = 0$ with

$$q = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \qquad f(q) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{bmatrix}$$

where $E = \rho e + \frac{1}{2}\rho u^2$

Equation of state: $p =$ pressure $= p(\rho, E)$

Ideal gas, polytropic EOS: $p = \rho e(\gamma - 1) = (\gamma - 1)\left(E - \frac{1}{2}\rho u^2\right)$

$\gamma \approx 7/5 = 1.4$ for air, $\gamma = 5/3$ for monatomic gas

The Jacobian $f'(q)$ has eigenvalues $u - c,\ \ u,\ \ u + c$ where

$$c = \sqrt{\left.\frac{dp}{d\rho}\right|_{\text{at constant entropy}}} = \sqrt{\frac{\gamma p}{\rho}} \text{ for ideal gas}$$

Riemann Problem for Euler equations

Initial data:

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$

Shock tube problem: $u_l = u_r = 0$, jump in ρ and p.

Pressure:

This is also solution to dam break problem for shallow water equations.

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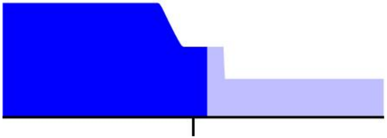

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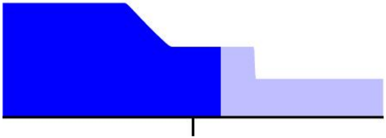

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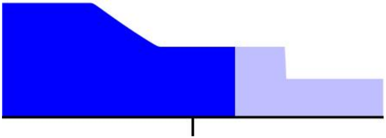

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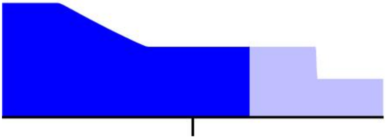

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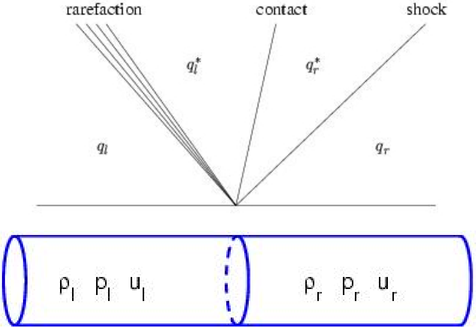
Riemann Problem for gas dynamics

Waves propagating in x–t space:



Similarity solution (function of x/t alone).

Waves can be approximated by discontinuities:

Approximate Riemann solvers