The Riemann problem

The Riemann problem consists of the hyperbolic equation under study together with initial data of the form

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x \geq 0 \end{cases}$$

Piecewise constant with a single jump discontinuity from q_l to q_r.

The Riemann problem is fundamental to understanding

- The mathematical theory of hyperbolic problems,
- Godunov-type finite volume methods

Why? Even for nonlinear systems of conservation laws, the Riemann problem can often be solved for general q_l and q_r, and consists of a set of waves propagating at constant speeds.

The Riemann problem for advection

The Riemann problem for the advection equation $q_t + uq_x = 0$ with

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x \ge 0 \end{cases}$$

has solution

$$q(x,t) = q(x-ut,0) = \begin{cases} q_l & \text{if } x < ut \\ q_r & \text{if } x \ge ut \end{cases}$$

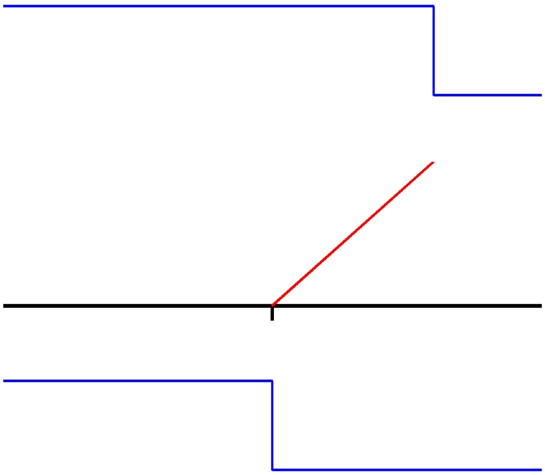
consisting of a single wave of strength $\mathcal{W}^1 = q_r - q_l$ propagating with speed $s^1 = u$.

Riemann solution for advection

$q(x, T)$

x–t plane

$q(x, 0)$



Discontinuous solutions

Note: The Riemann solution is not a classical solution of the PDE $q_t + uq_x = 0$, since q_t and q_x blow up at the discontinuity.

Integral form:

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,dx = uq(x_1,t) - uq(x_2,t)$$

Integrate in time from t_1 to t_2 to obtain

$$\int_{x_1}^{x_2} q(x,t_2)\,dx - \int_{x_1}^{x_2} q(x,t_1)\,dx$$
$$= \int_{t_1}^{t_2} uq(x_1,t)\,dt - \int_{t_1}^{t_2} uq(x_2,t)\,dt.$$

The Riemann solution satisfies the given initial conditions and this integral form for all $x_2 > x_1$ and $t_2 > t_1 \geq 0$.

Diffusive flux

$q(x,t) =$ concentration
$\beta =$ diffusion coefficient ($\beta > 0$)

diffusive flux $= -\beta q_x(x,t)$

$q_t + f_x = 0 \implies$ diffusion equation:

$$q_t = (\beta q_x)_x = \beta q_{xx} \text{ (if } \beta = \text{const).}$$

Diffusive flux

$q(x,t) =$ concentration
$\beta =$ diffusion coefficient ($\beta > 0$)

diffusive flux $= -\beta q_x(x,t)$

$q_t + f_x = 0 \implies$ diffusion equation:

$$q_t = (\beta q_x)_x = \beta q_{xx} \text{ (if } \beta = \text{const).}$$

Heat equation: Same form, where

$q(x,t) =$ density of thermal energy $= \kappa T(x,t)$,
$T(x,t) =$ temperature, $\kappa =$ heat capacity,
flux $= -\beta T(x,t) = -(\beta/\kappa) q(x,t) \implies$

$$q_t(x,t) = (\beta/\kappa) q_{xx}(x,t).$$

Advection-diffusion

$q(x,t) =$ concentration that advects with velocity u
and diffuses with coefficient β:

$$\text{flux} = uq - \beta q_x.$$

Advection-diffusion equation:

$$q_t + uq_x = \beta q_{xx}.$$

If $\beta > 0$ then this is a parabolic equation.

Advection dominated if u/β (the Péclet number) is large.

Fluid dynamics: "parabolic terms" arise from

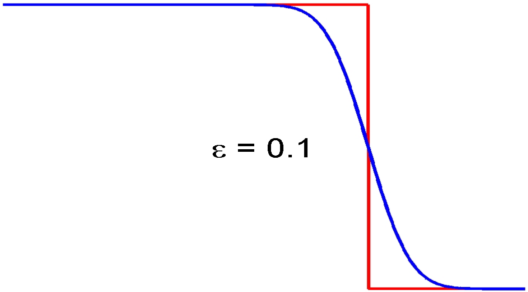
- thermal diffusion and
- diffusion of momentum, where the diffusion parameter is the viscosity.

Discontinuous solutions

Vanishing Viscosity solution: The Riemann solution $q(x,t)$ is the limit as $\epsilon \to 0$ of the solution $q^\epsilon(x,t)$ of the parabolic advection-diffusion equation

$$q_t + uq_x = \epsilon q_{xx}.$$

For any $\epsilon > 0$ this has a classical smooth solution:

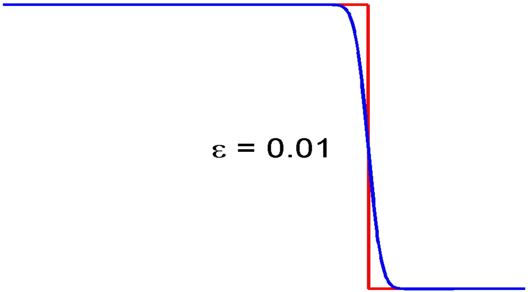


Discontinuous solutions

Vanishing Viscosity solution: The Riemann solution $q(x,t)$ is the limit as $\epsilon \to 0$ of the solution $q^\epsilon(x,t)$ of the parabolic advection-diffusion equation

$$q_t + uq_x = \epsilon q_{xx}.$$

For any $\epsilon > 0$ this has a classical smooth solution:

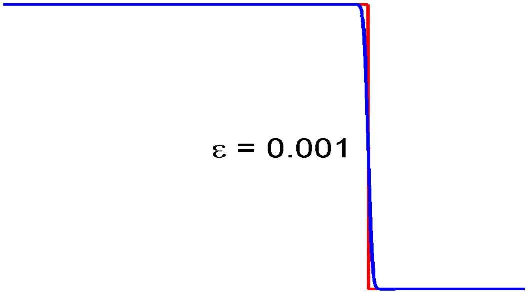


Discontinuous solutions

Vanishing Viscosity solution: The Riemann solution $q(x,t)$ is the limit as $\epsilon \to 0$ of the solution $q^\epsilon(x,t)$ of the parabolic advection-diffusion equation

$$q_t + uq_x = \epsilon q_{xx}.$$

For any $\epsilon > 0$ this has a classical smooth solution:



Nonlinear Burgers' equation

Conservation form: $u_t + \left(\frac{1}{2}u^2\right)_x = 0, \qquad f(u) = \frac{1}{2}u^2$.

Quasi-linear form: $\qquad u_t + uu_x = 0.$

Nonlinear Burgers' equation

Conservation form: $u_t + \left(\frac{1}{2}u^2\right)_x = 0, \qquad f(u) = \frac{1}{2}u^2$.

Quasi-linear form: $\qquad u_t + uu_x = 0.$

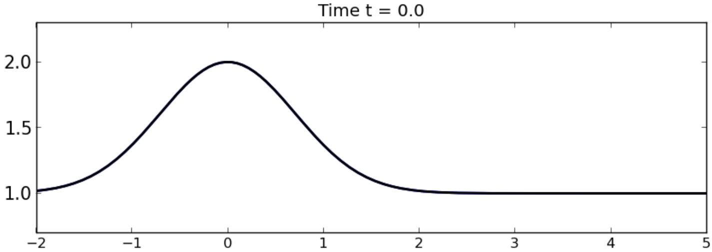
This looks like an advection equation with u advected with speed u.

True solution: u is constant along characteristic with speed $f'(u) = u$ until the wave "breaks" (shock forms).

Burgers' equation

Quasi-linear form: $u_t + uu_x = 0$

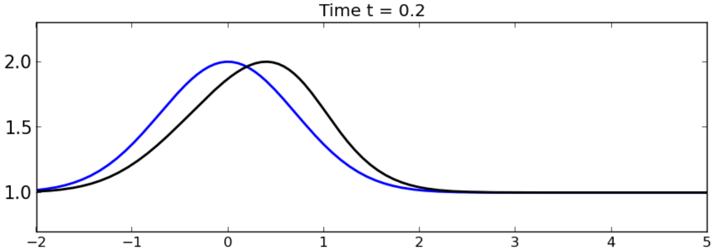
The solution is constant on characteristics so each value advects at constant speed equal to the value...



Burgers' equation

Quasi-linear form: $u_t + uu_x = 0$

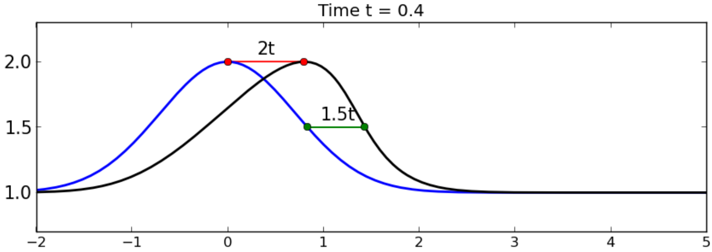
The solution is constant on characteristics so each value advects at constant speed equal to the value...



Burgers' equation

Quasi-linear form: $u_t + uu_x = 0$

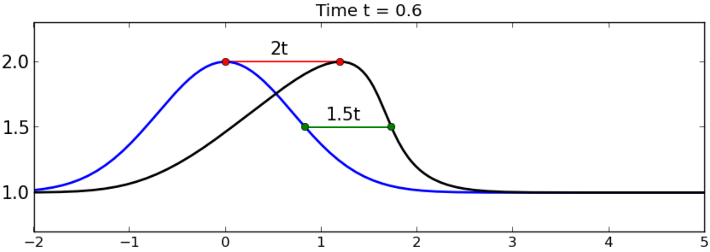
The solution is constant on characteristics so each value advects at constant speed equal to the value...



Burgers' equation

Quasi-linear form: $u_t + uu_x = 0$

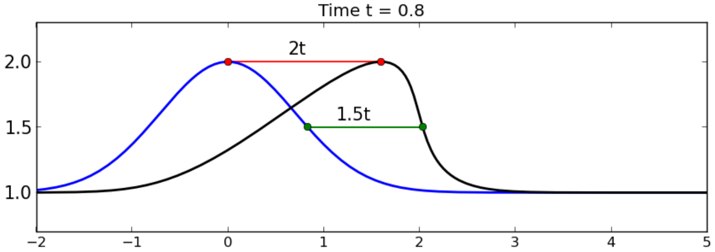
The solution is constant on characteristics so each value advects at constant speed equal to the value...



Burgers' equation

Quasi-linear form: $u_t + uu_x = 0$

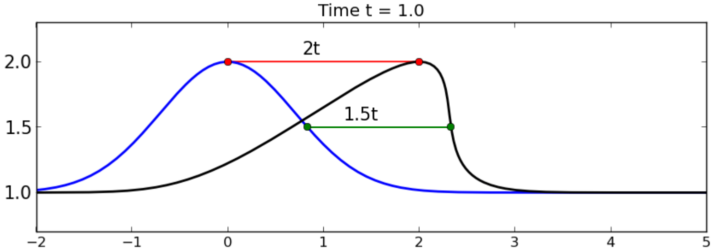
The solution is constant on characteristics so each value advects at constant speed equal to the value...



Burgers' equation

Quasi-linear form: $u_t + uu_x = 0$

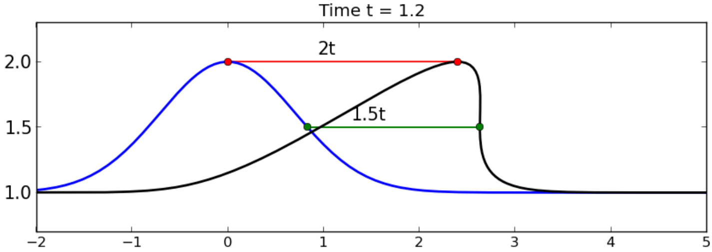
The solution is constant on characteristics so each value advects at constant speed equal to the value...



Burgers' equation

Quasi-linear form: $u_t + uu_x = 0$

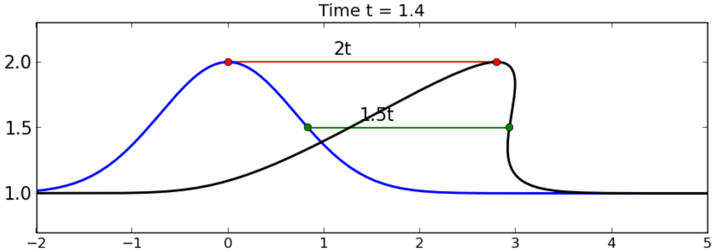
The solution is constant on characteristics so each value advects at constant speed equal to the value...



Burgers' equation

Quasi-linear form: $u_t + uu_x = 0$

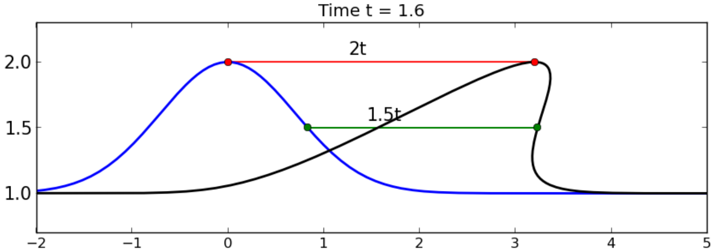
The solution is constant on characteristics so each value advects at constant speed equal to the value...



Burgers' equation

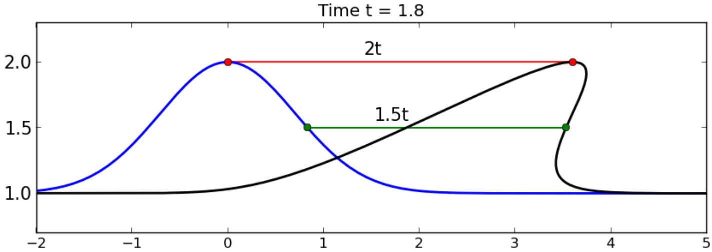
Quasi-linear form: $u_t + uu_x = 0$

The solution is constant on characteristics so each value advects at constant speed equal to the value...



Quasi-linear form: $u_t + uu_x = 0$

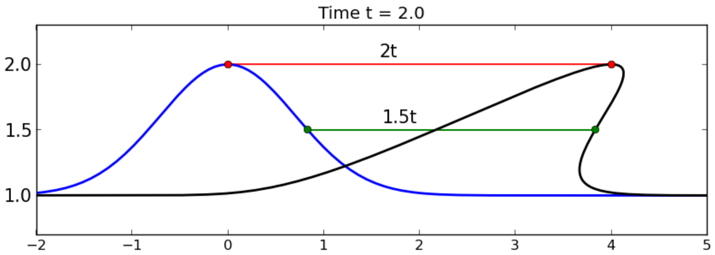
The solution is constant on characteristics so each value advects at constant speed equal to the value...



Burgers' equation

Quasi-linear form: $u_t + uu_x = 0$

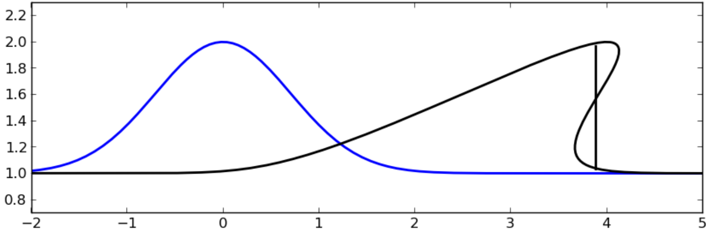
The solution is constant on characteristics so each value advects at constant speed equal to the value...



Equal-area rule:

The area "under" the curve is conserved with time,

We must insert a shock so the two areas cut off are equal.



Vanishing Viscosity solution

Viscous Burgers' equation: $u_t + \left(\frac{1}{2}u^2\right)_x = \epsilon u_{xx}.$

This parabolic equation has a smooth C^∞ solution for all $t > 0$ for any initial data.

Vanishing Viscosity solution

Viscous Burgers' equation: $u_t + \left(\frac{1}{2}u^2\right)_x = \epsilon u_{xx}$.

This parabolic equation has a smooth C^∞ solution for all $t > 0$ for any initial data.

Limiting solution as $\epsilon \to 0$ gives the shock-wave solution.

Vanishing Viscosity solution

Viscous Burgers' equation: $u_t + \left(\frac{1}{2}u^2\right)_x = \epsilon u_{xx}$.

This parabolic equation has a smooth C^∞ solution for all $t > 0$ for any initial data.

Limiting solution as $\epsilon \to 0$ gives the shock-wave solution.

Why try to solve hyperbolic equation?

Vanishing Viscosity solution

Viscous Burgers' equation: $u_t + \left(\frac{1}{2}u^2\right)_x = \epsilon u_{xx}$.

This parabolic equation has a smooth C^∞ solution for all $t > 0$ for any initial data.

Limiting solution as $\epsilon \to 0$ gives the shock-wave solution.

Why try to solve hyperbolic equation?

- Solving parabolic equation requires implicit method,
- Often correct value of physical "viscosity" is very small, shock profile that cannot be resolved on the desired grid
  $\Longrightarrow$ smoothness of exact solution doesn't help!