AMath 574 January 31, 2011

Today:

- Boundary conditions
- Multi-dimensional

Wednesday and Friday:

- More multi-dimensional

Reading: Chapters 18, 19, 20

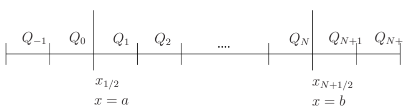
Notes:

R.J. LeVeque, University of Washington AMath 574, January 31, 2011

Boundary conditions and ghost cells

In each time step, the data in cells 1 to N is used to define ghost cell values in cells outside the physical domain.

The wave-propagation algorithm is then applied on the expanded computational domain, solving Riemann problems at all interfaces.



The data is extended depending on the physical boundary conditons.

Notes:

R.J. LeVeque, University of Washington AMath 574, January 31, 2011 [FVMHP Chap. 7]

Initial–boundary value problem (IBVP) for advection

Advection equation on finite 1D domain:

$$q_t + uq_x = 0 \qquad a < x < b, \;\; t \geq 0,$$

with initial data

$$q(x, 0) = \eta(x) \qquad a < x < b.$$

and boundary data at the inflow boundary:

If $u > 0$, need data at $x = a$:

$$q(a, t) = g(t), \qquad t \geq 0,$$

If $u < 0$, need data at $x = b$:

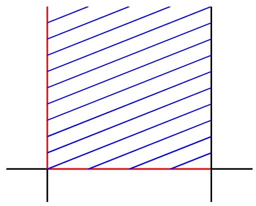
$$q(b, t) = g(t), \qquad t \geq 0,$$

Notes:

R.J. LeVeque, University of Washington AMath 574, January 31, 2011 [FVMHP Sec. 2.1]

Characteristics for IBVP

In x–t plane for the case $u > 0$:



Solution:

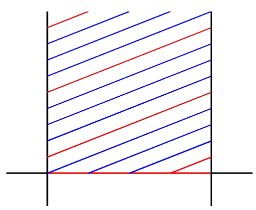
$$q(x,t) = \begin{cases} \eta(x - ut) & \text{if } a \le x - ut \le b, \\ g((x-a)/u) & \text{otherwise} \ . \end{cases}$$

Notes:

Periodic boundary conditions

$q(a,t) = q(b,t), \qquad t \ge 0.$

In x–t plane for the case $u > 0$:



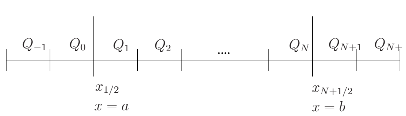
Solution:

$$q(x,t) = \eta(X_0(x,t)),$$

where $X_0(x,t) = a + \text{mod}(x - ut - a,\ b - a)$.

Notes:

Boundary conditions



Periodic:

$$Q^n_{-1} = Q^n_{N-1}, \quad Q^n_0 = Q^n_N, \quad Q^n_{N+1} = Q^n_1, \quad Q^n_{N+2} = Q^n_2$$

Extrapolation (outflow):

$$Q^n_{-1} = Q^n_1, \quad Q^n_0 = Q^n_1, \quad Q^n_{N+1} = Q^n_N, \quad Q^n_{N+2} = Q^n_N$$

Solid wall:

For Q_0 : $\quad p_0 = p_1, \quad u_0 = -u_1,$

For Q_{-1} : $\quad p_{-1} = p_2, \quad u_{-1} = -u_2.$

Notes:

Extrapolation boundary conditions

If we set $Q_0 = Q_1$ then the Riemann problem at $x_{1/2}$ has zero strength waves:

$$Q_1 - Q_0 = \mathcal{W}^1_{1/2} + \mathcal{W}^2_{1/2}$$

So in particular the incoming wave $\mathcal{W}^2$ has strength 0.

The outgoing wave perhaps should have nonzero magnitude, but it doesn't matter since it would only update ghost cell.

Ghost cell value is reset at the start of each time step by extrapolation.

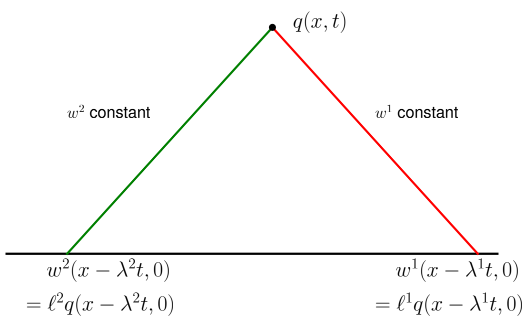
In 2D or 3D, extrapolation in normal direction is not perfect but works quite well, e.g. Figure 21.7.

Notes:

Solution by tracing back on characteristics

The general solution for acoustics:

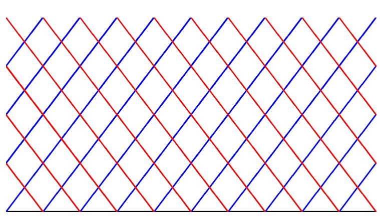
$$q(x,t) = w^1(x - \lambda^1 t, 0) r^1 + w^2(x - \lambda^2 t, 0) r^2$$



Notes:

Linear acoustics — characteristics

$$\begin{aligned} q(x,t) &= w^1(x+ct,0)r^1 + w^2(x-ct,0)r^2 \\ &= \frac{-p_0(x+ct)}{2Z}\begin{bmatrix} -Z \\ 1 \end{bmatrix} + \frac{p_0(x-ct)}{2Z}\begin{bmatrix} Z \\ 1 \end{bmatrix}. \end{aligned}$$



For IBVP on $a < x < b$, must specify one incoming boundary condition at each side: $w^2(a,t)$ and $w^1(b,t)$

Notes:

Acoustics boundary conditions

Non-reflecting boundary conditions:

$$w^2(a,t) = 0, \qquad w^1(b,t) = 0.$$

Periodic boundary conditions:

$$w^2(a,t) = w^2(b,t), \qquad w^1(b,t) = w^1(a,t),$$

or simply

$$q(a,t) = q(b,t).$$

Solid wall (reflecting) boundary conditions:

$$u(a,t) = 0, \qquad u(b,t) = 0.$$

which can be written in terms of characteristic variables as:

$$w^2(a,t) = -w^1(a,t), \qquad w^1(b,t) = -w^2(a,t)$$

since $u = w^1 + w^2$.

Notes:

Zero velocity (solid wall) boundary condition

For acoustics $q = (p, u)$ or Euler with $q = (\rho,\ \rho u,\ E)$.

To obtain $u(0,t) = 0$, set

$u_0 = -u_1$, and extrapolate other values, e.g. $p_0 = p_1$

Then Riemann solution is a similarity solution $q(x,t) = Q^*(x/t)$ and $u(0) = 0$ (i.e., middle state has $u_m = 0$ for acoustics).

Notes:

Incoming wave train

Acoustics with right-going wave train coming from $x = -\infty$.

$$p(0,t) = \bar{p}\sin(kt), \quad u(0,t) = (\bar{p}/Z)\sin(kt)$$

If wave reflects in domain then there will also be an outgoing wave.

We want the BC to give this incoming BC and be non-reflecting. We need to specify

$$\begin{aligned} Q_0^n &= \begin{bmatrix} p(-\Delta x/2, t_n) \\ u(-\Delta x/2, t_n) \end{bmatrix} \\ &= w^1 \begin{bmatrix} -Z \\ 1 \end{bmatrix} + w^2(-\Delta x/2, t_n) \begin{bmatrix} Z \\ 1 \end{bmatrix} \\ &= w^1 \begin{bmatrix} -Z \\ 1 \end{bmatrix} + w^2(0, t_n + \Delta x/(2c)) \begin{bmatrix} Z \\ 1 \end{bmatrix} \end{aligned}$$

Notes:

Incoming wave train

$$\begin{aligned} Q_0^n &= \begin{bmatrix} p(-\Delta x/2, t_n) \\ u(-\Delta x/2, t_n) \end{bmatrix} \\ &= w^1 \begin{bmatrix} -Z \\ 1 \end{bmatrix} + w^2(0, t_n + \Delta x/(2c)) \begin{bmatrix} Z \\ 1 \end{bmatrix} \end{aligned}$$

The first term is eliminated by setting $w^1 = 0$
(doesn't matter what it is since ghost cell reset).

Since $w^2 = \frac{1}{2Z}(p + Zu)$ we are left with

$$Q_0 = \bar{p} \sin(k(t_n + \Delta x/(2c))) \begin{bmatrix} 1 \\ 1/Z \end{bmatrix}.$$

Similarly,

$$Q_{-1} = \bar{p} \sin(k(t_n + 3\Delta x/(2c))) \begin{bmatrix} 1 \\ 1/Z \end{bmatrix}.$$

Notes:

First order hyperbolic PDE in 2 space dimensions

Advection equation: $q_t + uq_x + vq_y = 0$

First-order system: $q_t + Aq_x + Bq_y = 0$

where $q \in \mathbb{R}^m$ and $A, B \in \mathbb{R}^{m \times m}$.

Hyperbolic if $\cos(\theta)A + \sin(\theta)B$ is diagonalizable with real eigenvalues, for all angles θ.

This is required so that plane-wave data gives a 1d hyperbolic problem:

$$q(x, y, 0) = \breve{q}(x \cos\theta + y \sin\theta)$$ (\breve q)

implies contours of q in x–y plane are orthogonal to θ–direction.

Notes:

Plane wave solutions

Suppose

$$\begin{aligned} q(x, y, t) &= \breve{q}(x \cos\theta + y \sin\theta, t) \\ &= \breve{q}(\xi, t). \end{aligned}$$

Then:

$$\begin{aligned} q_x(x, y, t) &= \cos\theta \, \breve{q}_\xi(\xi, t) \\ q_y(x, y, t) &= \sin\theta \, \breve{q}_\xi(\xi, t) \end{aligned}$$

so

$$q_t + Aq_x + Bq_y = \breve{q}_t + (A\cos\theta + B\sin\theta)\breve{q}_\xi$$

and the 2d problem reduces to the 1d hyperbolic equation

$$\breve{q}_t(\xi, t) + (A\cos\theta + B\sin\theta)\breve{q}_\xi(\xi, t) = 0.$$

Notes:

Advection in 2 dimensions

Constant coefficient: $q_t + uq_x + vq_y = 0$

In this case solution for arbitrary initial data is easy:

$$q(x, y, t) = q(x - ut,\ y - vt,\ 0).$$

Data simply shifts at constant velocity (u, v) in x-y plane.

Variable coefficient:

Conservation form: $q_t + (u(x, y, t)q)_x + (v(x, y, t)q)_y = 0$

Advective form (color eqn): $q_t + u(x, y, t)q_x + v(x, y, t)q_y = 0$

Equivalent only if flow is divergence-free (incompressible):

$$\nabla \cdot \vec{u} = u_x(x, y, t) + v_y(x, y, t) = 0 \qquad \forall t \geq 0.$$

Notes:

R.J. LeVeque, University of Washington AMath 574, January 31, 2011 [FVMHP Chap. 18]

Advection in 2 dimensions: characteristics

The characteristic curve $(X(t), Y(t))$ starting at some (x_0, y_0) is determined by solving the ODEs

$$X'(t) = u(X(t),\ Y(t),\ t), \qquad X(0) = x_0$$
$$Y'(t) = v(X(t),\ Y(t),\ t), \qquad Y(0) = y_0.$$

How does q vary along this curve?

$$\frac{\partial}{\partial t} q(X(t),\ Y(t),\ t) = X'(t)q_x(\cdots) + Y'(t)q_y(\cdots) + q_t(\cdots)$$

For color equation: $q_t + u(x, y, t)q_x + v(x, y, t)q_y = 0$

q is constant along characterisitic (color is advected).

Notes:

R.J. LeVeque, University of Washington AMath 574, January 31, 2011 [FVMHP Chap. 18]

Advection in 2 dimensions: characteristics

For conservative equation: $q_t + (u(x, y, t)q)_x + (v(x, y, t)q)_y = 0$

Can rewrite as $q_t + u(x, y, t)q_x + v(x, y, t)q_y = (u_x + v_y)q$

Along characteristic q varies because of source term:

$$\frac{\partial}{\partial t} q(X(t),\ Y(t),\ t) = X'(t)q_x(\cdots) + Y'(t)q_y(\cdots) + q_t(\cdots)$$
$$= (\nabla \cdot \vec{u})q.$$

Conservative form models density of conserved quantity.

Mass in region advecting with the flow varies stays constant
but density increases if volume of region decreases,
or density decreases if volume of region increases.

Notes:

R.J. LeVeque, University of Washington AMath 574, January 31, 2011 [FVMHP Chap. 18]

Acoustics in 2 dimensions

$$p_t + K(u_x + v_y) = 0$$
$$\rho u_t + p_x = 0$$
$$\rho v_t + p_y = 0$$

Note: pressure responds to compression or expansion and so p_t is proportional to divergence of velocity.

Second and third equations are $F = ma$.

Gives hyperbolic system $q_t + Aq_x + Bq_y = 0$ with

$$q = \begin{bmatrix} p \\ u \\ v \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & K & 0 \\ 1/\rho & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & K \\ 0 & 0 & 0 \\ 1/\rho & 0 & 0 \end{bmatrix}.$$

Notes:

Acoustics in 2 dimensions

$$q = \begin{bmatrix} p \\ u \\ v \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & K & 0 \\ 1/\rho & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & K \\ 0 & 0 & 0 \\ 1/\rho & 0 & 0 \end{bmatrix}.$$

Plane waves:

$$A\cos\theta + B\sin\theta = \begin{bmatrix} 0 & K\cos\theta & K\sin\theta \\ \cos\theta/\rho & 0 & 0 \\ \sin\theta/\rho & 0 & 0 \end{bmatrix}.$$

Eigenvalues: $\lambda^1 = -c, \quad \lambda^2 = 0, \quad \lambda^3 = +c$ where $c = \sqrt{K/\rho}$

Independent of angle θ.

Isotropic: sound propagates at same speed in any direction.

Note: Zero wave speed for "shear wave" with variation only in velocity in direction $(-\sin\theta,\ \cos\theta)$. (Fig 18.1)

Notes:

Diagonalization 2 dimensions

Can we diagonalize system $q_t + Aq_x + Bq_y = 0$?

Only if A and B have the same eigenvectors!

If $A = R\Lambda R^{-1}$ and $B = RMR^{-1}$, then let $w = R^{-1}q$ and

$$w_t + \Lambda w_x + M w_y = 0$$

This decouples into scalar advection equations for each component of w:

$$w_t^p + \lambda^p w_x^p + \mu^p w_y^p = 0 \implies w^p(x,y,t) = w^p(x - \lambda^p t,\ y - \mu^p t,\ 0).$$

Note: In this case information propagates only in a finite number of directions $(\lambda^p,\ \mu^p)$ for $p = 1,\ \ldots,\ m$.

This is not true for most coupled systems, e.g. acoustics.

Notes: