Conservation Laws and Finite Volume Methods

AMath 574
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Notes:

Outline

Today:

- Finite volume methods
- Conservation form
- Godunov's method
- Upwind method for advection, linear system
- CFL condition

Next:

- High resolution methods

Reading: Chapters 5 and 6

Notes:

Finite differences vs. finite volumes

Finite difference Methods

- Pointwise values $Q_i^n \approx q(x_i, t_n)$
- Approximate derivatives by finite differences
- Assumes smoothness

Finite volume Methods

- Approximate cell averages: $Q_i^n \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t_n)\, dx$
- Integral form of conservation law,

$$\frac{\partial}{\partial t} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)\, dx \quad = \quad f(q(x_{i-1/2}, t)) - f(q(x_{i+1/2}, t))$$

leads to conservation law $q_t + f_x = 0$ but also directly to numerical method.

Notes:

Finite volume method

Based on cell averages:

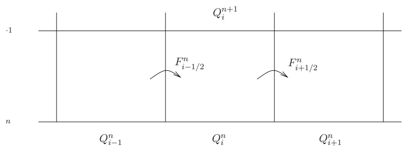
$$Q_i^n \approx \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t_n)\,dx$$

Update cell average by flux into and out of cell:

Ex: Upwind methods for advection equation $q_t + uq_x = 0$:

$$\begin{aligned} Q_i^{n+1} &= Q_i^n - \frac{\Delta t(uQ_{i-1}^n - uQ_i^n)}{\Delta x} \\ &= Q_i^n - \frac{\Delta t u}{\Delta x}(Q_i^n - Q_{i-1}^n) \end{aligned}$$

Stencil:
(x-t plane)



Notes:

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Nonlinear scalar conservation laws

Burgers' equation: $u_t + \left(\frac{1}{2}u^2\right)_x = 0$.

Quasilinear form: $u_t + uu_x = 0$.

These are equivalent for smooth solutions, not for shocks!

Upwind methods for $u > 0$:

Conservative: $U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}((U_i^n)^2 - (U_{i-1}^n)^2)\right)$

Quasilinear: $U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}U_i^n(U_i^n - U_{i-1}^n)$.

Ok for smooth solutions, not for shocks!

Notes:

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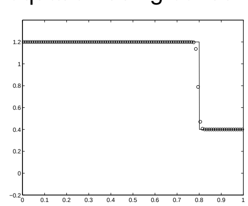
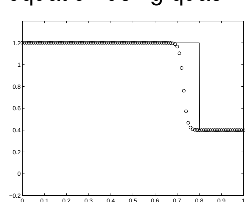
Importance of conservation form

Solution to Burgers' equation using conservative upwind:



Solution to Burgers' equation using quasilinear upwind:



Notes:

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Conservation form

The method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n)$$

is in conservation form.

The total mass is conserved up to fluxes at the boundaries:

$$\Delta x \sum_i Q_i^{n+1} = \Delta x \sum_i Q_i^n - \frac{\Delta t}{\Delta x}(F_{+\infty} - F_{-\infty}).$$

Note: an isolated shock must travel at the right speed!

Notes:

Lax-Wendroff Theorem

Suppose the method is conservative and consistent with $q_t + f(q)_x = 0$,

$$F_{i-1/2} = \mathcal{F}(Q_{i-1}, Q_i) \quad \text{with } \mathcal{F}(\bar{q}, \bar{q}) = f(\bar{q})$$

and Lipschitz continuity of $\mathcal{F}$.

If a sequence of discrete approximations converge to a function $q(x,t)$ as the grid is refined, then this function is a weak solution of the conservation law.

Note:

Does not guarantee a sequence converges

Two sequences might converge to different weak solutions.

Also need stability and entropy condition.

Notes:

Finite volume method

Based on cell averages:

$$Q_i^n \approx \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t_n)\,dx$$

Update cell average by flux into and out of cell:

Ex: Upwind methods for advection equation $q_t + uq_x = 0$:

$$\begin{aligned} Q_i^{n+1} &= Q_i^n - \frac{\Delta t(uQ_{i-1}^n - uQ_i^n)}{\Delta x} \\ &= Q_i^n - \frac{\Delta t u}{\Delta x}(Q_i^n - Q_{i-1}^n) \end{aligned}$$

Stencil:
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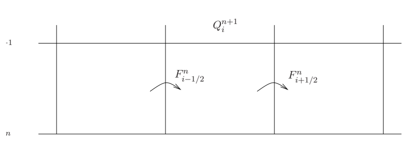
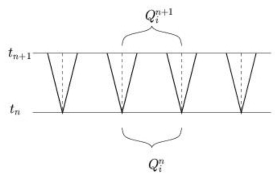


Notes:

Godunov's Method for $q_t + f(q)_x = 0$

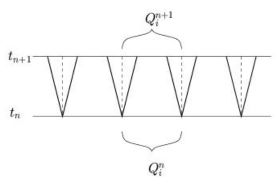


1. Solve Riemann problems at all interfaces, yielding waves $\mathcal{W}_{i-1/2}^p$ and speeds $s_{i-1/2}^p$, for $p = 1,\ 2,\ \ldots,\ m$.

Riemann problem: Original equation with piecewise constant data.

Notes:

Godunov's Method for $q_t + f(q)_x = 0$



Then either:

1. Compute new cell averages by integrating over cell at t_{n+1},
2. Compute fluxes at interfaces and flux-difference:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[F_{i+1/2}^n - F_{i-1/2}^n]$$

3. Update cell averages by contributions from all waves entering cell:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}]$$

where $\mathcal{A}^\pm \Delta Q_{i-1/2} = \sum_{i=1}^{m} (s_{i-1/2}^p)^\pm \mathcal{W}_{i-1/2}^p$.

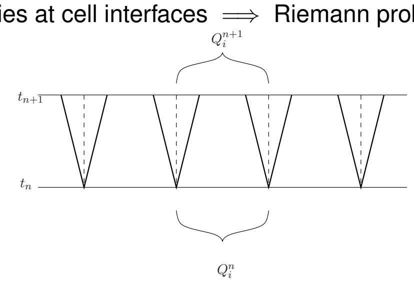
Notes:

Godunov's method

Q_i^n defines a piecewise constant function

$$\tilde{q}^n(x, t_n) = Q_i^n \;\text{ for } x_{i-1/2} < x < x_{i+1/2}$$

Discontinuities at cell interfaces $\Longrightarrow$ Riemann problems.



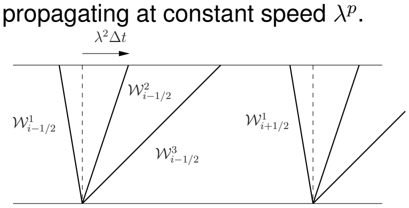
$\tilde{q}^n(x_{i-1/2}, t) \equiv q^{\Downarrow}(Q_{i-1}, Q_i)$ for $t > t_n$.

$$F_{i-1/2}^n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q^{\Downarrow}(Q_{i-1}^n, Q_i^n))\,dt = f(q^{\Downarrow}(Q_{i-1}^n, Q_i^n)).$$

Notes:

Wave-propagation viewpoint

For linear system $q_t + Aq_x = 0$, the Riemann solution consists of waves $\mathcal{W}^p$ propagating at constant speed λ^p.



$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha^p_{i-1/2} r^p \equiv \sum_{p=1}^{m} \mathcal{W}^p_{i-1/2}.$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\lambda^2 \mathcal{W}^2_{i-1/2} + \lambda^3 \mathcal{W}^3_{i-1/2} + \lambda^1 \mathcal{W}^1_{i+1/2}\right].$$

Notes:

First-order REA Algorithm

1. Reconstruct a piecewise constant function $\tilde{q}^n(x, t_n)$ defined for all x, from the cell averages Q_i^n.

$$\tilde{q}^n(x, t_n) = Q_i^n \quad \text{for all } x \in \mathcal{C}_i.$$

2. Evolve the hyperbolic equation exactly (or approximately) with this initial data to obtain $\tilde{q}^n(x, t_{n+1})$ a time Δt later.

3. Average this function over each grid cell to obtain new cell averages

$$Q_i^{n+1} = \frac{1}{\Delta x}\int_{\mathcal{C}_i} \tilde{q}^n(x, t_{n+1})\,dx.$$

Notes:

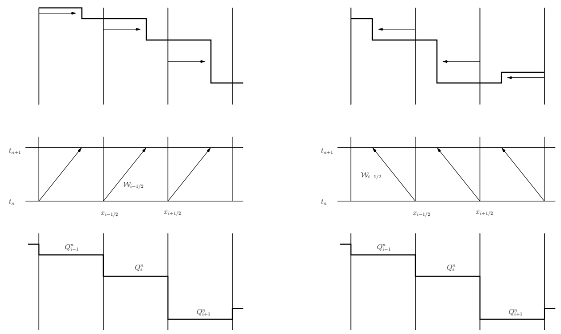
Godunov's method for advection

Q_i^n defines a piecewise constant function

$$\tilde{q}^n(x, t_n) = Q_i^n \text{ for } x_{i-1/2} < x < x_{i+1/2}$$

Discontinuities at cell interfaces $\Longrightarrow$ Riemann problems.

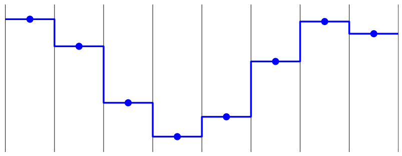
$u > 0$ $u < 0$



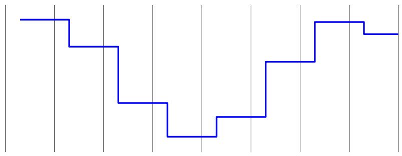
Notes:

First-order REA Algorithm

Cell averages and piecewise constant reconstruction:

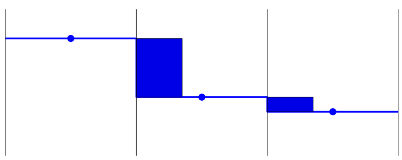


After evolution:



Notes:

Cell update



The cell average is modified by

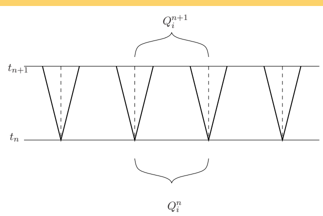
$$\frac{u\Delta t \cdot (Q_{i-1}^n - Q_i^n)}{\Delta x}$$

So we obtain the upwind method

$$Q_i^{n+1} = Q_i^n - \frac{u\Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n).$$

Notes:

Godunov (upwind) on acoustics



Data at time t_n : $\tilde{q}^n(x, t_n) = Q_i^n$ for $x_{i-1/2} < x < x_{i+1/2}$
Solving Riemann problems for small Δt gives solution:

$$\tilde{q}^n(x, t_{n+1}) = \begin{cases} Q_{i-1/2}^* & \text{if } x_{i-1/2} - c\Delta t < x < x_{i-1/2} + c\Delta t, \\ Q_i^n & \text{if } x_{i-1/2} + c\Delta t < x < x_{i+1/2} - c\Delta t, \\ Q_{i+1/2}^* & \text{if } x_{i+1/2} - c\Delta t < x < x_{i+1/2} + c\Delta t, \end{cases}$$

So computing cell average gives:

$$Q_i^{n+1} = \frac{1}{\Delta x}\left[c\Delta t Q_{i-1/2}^* + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t Q_{i+1/2}^*\right].$$

Notes:

Godunov (upwind) on acoustics

$$Q_i^{n+1} = \frac{1}{\Delta x}\left[c\Delta t Q^*_{i-1/2} + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t Q^*_{i+1/2}\right].$$

Solve Riemann problems:

$$Q_i^n - Q_{i-1}^n = \Delta Q_{i-1/2} = \mathcal{W}^1_{i-1/2} + \mathcal{W}^2_{i-1/2} = \alpha^1_{i-1/2}r^1 + \alpha^2_{i-1/2}r^2,$$
$$Q_{i+1}^n - Q_i^n = \Delta Q_{i+1/2} = \mathcal{W}^1_{i+1/2} + \mathcal{W}^2_{i+1/2} = \alpha^1_{i+1/2}r^1 + \alpha^2_{i+1/2}r^2,$$

The intermediate states are:

$$Q^*_{i-1/2} = Q_i^n - \mathcal{W}^2_{i-1/2}, \qquad Q^*_{i+1/2} = Q_i^n + \mathcal{W}^1_{i+1/2},$$

So,

$$Q_i^{n+1} = \frac{1}{\Delta x}\left[c\Delta t(Q_i^n - \mathcal{W}^2_{i-1/2}) + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t(Q_i^n + \mathcal{W}^1_{i+1/2})\right]$$
$$= Q_i^n - \frac{c\Delta t}{\Delta x}\mathcal{W}^2_{i-1/2} + \frac{c\Delta t}{\Delta x}\mathcal{W}^1_{i+1/2}.$$

Notes:

Godunov (upwind) on acoustics

$$Q_i^{n+1} = \frac{1}{\Delta x}\left[c\Delta t Q^*_{i-1/2} + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t Q^*_{i+1/2}\right]$$
$$= \frac{1}{\Delta x}\left[c\Delta t(Q_i^n - \mathcal{W}^2_{i-1/2}) + (\Delta x - 2c\Delta t)Q_i^n + c\Delta t(Q_i^n + \mathcal{W}^1_{i+1/2})\right]$$
$$= Q_i^n - \frac{c\Delta t}{\Delta x}\mathcal{W}^2_{i-1/2} + \frac{c\Delta t}{\Delta x}\mathcal{W}^1_{i+1/2}$$
$$= Q_i^n - \frac{\Delta t}{\Delta x}(c\mathcal{W}^2_{i-1/2} + (-c)\mathcal{W}^1_{i+1/2}).$$

General form for linear system with m equations:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\sum_{p:\lambda^p>0}\lambda^p\mathcal{W}^p_{i-1/2} + \sum_{p:\lambda^p<0}\lambda^p\mathcal{W}^p_{i+1/2}\right]$$
$$= Q_i^n - \frac{\Delta t}{\Delta x}\left[\sum_{m=1}^{p}(\lambda^p)^+\mathcal{W}^p_{i-1/2} + \sum_{m=1}^{p}(\lambda^p)^-\mathcal{W}^p_{i+1/2}\right]$$

Notes:

Godunov (upwind) on acoustics

Solve Riemann problems:

$$Q_i^n - Q_{i-1}^n = \Delta Q_{i-1/2} = \mathcal{W}^1_{i-1/2} + \mathcal{W}^2_{i-1/2} = \alpha^1_{i-1/2}r^1 + \alpha^2_{i-1/2}r^2,$$
$$Q_{i+1}^n - Q_i^n = \Delta Q_{i+1/2} = \mathcal{W}^1_{i+1/2} + \mathcal{W}^2_{i+1/2} = \alpha^1_{i+1/2}r^1 + \alpha^2_{i+1/2}r^2,$$

The waves are determined by solving for α from $R\alpha = \Delta Q$:

$$A = \begin{bmatrix} 0 & K \\ 1/\rho & 0 \end{bmatrix}, \qquad R = \begin{bmatrix} -Z & Z \\ 1 & 1 \end{bmatrix}, \qquad R^{-1} = \frac{1}{2Z}\begin{bmatrix} -1 & Z \\ 1 & Z \end{bmatrix}.$$

So

$$\Delta Q = \begin{bmatrix} \Delta p \\ \Delta u \end{bmatrix} = \alpha^1\begin{bmatrix} -Z \\ 1 \end{bmatrix} + \alpha^2\begin{bmatrix} Z \\ 1 \end{bmatrix}$$

with

$$\alpha^1 = \frac{1}{2Z}(-\Delta p + Z\Delta u), \qquad \alpha^2 = \frac{1}{2Z}(\Delta p + Z\Delta u).$$

Notes:

Matrix splitting

Recall $A = R\Lambda R^{-1}$ with $\Lambda = \begin{bmatrix} -c & 0 \\ 0 & c \end{bmatrix}$.

Let

$$\Lambda^+ = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}, \qquad \Lambda^- = \begin{bmatrix} -c & 0 \\ 0 & 0 \end{bmatrix}.$$

and

$$A^+ = R\Lambda^+ R^{-1}, \qquad A^- = R\Lambda^- R^{-1}.$$

Then $A^+ + A^- = R(\Lambda^+ + \Lambda^-)R^{-1} = R\Lambda R^{-1} = A.$

$$\begin{aligned} A^+\Delta Q &= R\Lambda^+ R^{-1}\Delta Q = R\Lambda^+\alpha \\ &= \sum_{p=1}^m (\lambda^p)^+ \alpha^p r^p \end{aligned}$$

and similarly, $A^-\Delta Q = \sum_{p=1}^m (\lambda^p)^- \alpha^p r^p$

Notes:

Matrix splitting for upwind method

For $q_t + Aq_x = 0$, the upwind method (Godunov) is:

$$\begin{aligned} Q_i^{n+1} &= Q_i^n + \frac{\Delta t}{\Delta x}\left[\sum_{p=1}^m (\lambda^p)^+ \alpha_{i-1/2}^p r^p + \sum_{p=1}^m (\lambda^p)^- \alpha_{i+1/2}^p r^p\right] \\ &= Q_i^n + \frac{\Delta t}{\Delta x}\left[A^+\Delta Q_{i-1/2} + A^-\Delta Q_{i+1/2}\right] \\ &= Q_i^n + \frac{\Delta t}{\Delta x}\left[A^+(Q_i^n - Q_{i-1}^n) + A^-(Q_{i+1}^n - Q_i^n)\right] \end{aligned}$$

Natural generalization of upwind to a system.

If all eigenvalues are positive, then $A^+ = A$ and $A^- = 0$,

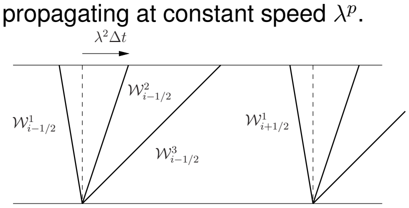
If all eigenvalues are negative, then $A^+ = 0$ and $A^- = A$.

Notes:

Wave-propagation viewpoint

For linear system $q_t + Aq_x = 0$, the Riemann solution consists of

waves $\mathcal{W}^p$ propagating at constant speed λ^p.



$$Q_i - Q_{i-1} = \sum_{p=1}^m \alpha_{i-1/2}^p r^p \equiv \sum_{p=1}^m \mathcal{W}_{i-1/2}^p.$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\lambda^2\mathcal{W}_{i-1/2}^2 + \lambda^3\mathcal{W}_{i-1/2}^3 + \lambda^1\mathcal{W}_{i+1/2}^1\right].$$

Notes:

The CFL Condition

Domain of dependence: The solution $q(X,T)$ depends on the data $q(x,0)$ over some set of x values, $x \in \mathcal{D}(X,T)$.

Advection: $q(X,T) = q(X - uT, 0)$ and so $\mathcal{D}(X,T) = \{X - uT\}$.

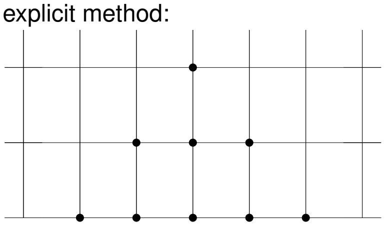
The CFL Condition: A numerical method can be convergent only if its numerical domain of dependence contains the true domain of dependence of the PDE, at least in the limit as Δt and Δx go to zero.

Note: Necessary but not sufficient for stability!

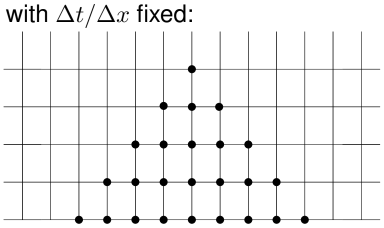
Notes:

Numerical domain of dependence

With a 3-point explicit method:



On a finer grid with $\Delta t / \Delta x$ fixed:



Notes:

The CFL Condition

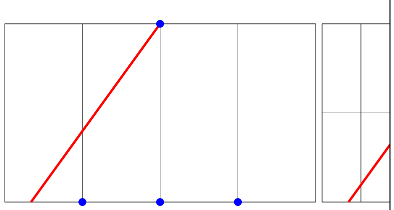
For the method to be stable, the numerical domain of dependence must include the true domain of dependence.

For advection, the solution is constant along characteristics,

$$q(x,t) = q(x - ut, 0)$$

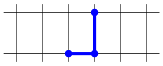
For a 3-point method, CFL condition requires $\left|\frac{u\Delta t}{\Delta x}\right| \le 1.$

If this is violated:
True solution is determined by data at a point $x - ut$ that is ignored by the numerical method, even as the grid is refined.

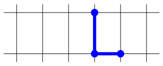


Notes:

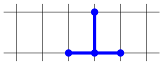
Stencil — CFL Condition



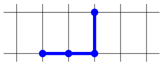
$$0 \le \frac{u\Delta t}{\Delta x} \le 1$$



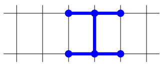
$$-1 \le \frac{u\Delta t}{\Delta x} \le 0$$



$$-1 \le \frac{u\Delta t}{\Delta x} \le 1$$



$$0 \le \frac{u\Delta t}{\Delta x} \le 2$$



$$-\infty < \frac{u\Delta t}{\Delta x} < \infty$$

Notes:

Linear hyperbolic systems

Linear system of m equations: $q(x,t) \in \mathbb{R}^m$ for each (x,t) and

$$q_t + Aq_x = 0, \qquad -\infty < x, \infty, \ \ t \ge 0.$$

A is $m \times m$ with eigenvalues λ^p and eigenvectors r^p, for $p = 1,\ 2,\ ,\dots,\ m$:

$$Ar^p = \lambda^p r^p.$$

Combining these for $p = 1,\ 2,\ ,\dots,\ m$ gives

$$AR = R\Lambda$$

where

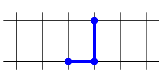
$$R = [r^1 \ \ r^2 \ \ \cdots \ \ r^m], \qquad \Lambda = \text{diag}(\lambda^1,\ \lambda^2,\ \dots,\ \lambda^m).$$

The system is hyperbolic if the eigenvalues are real and R is invertible. Then A can be diagonalized:

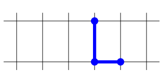
$$R^{-1}AR = \Lambda$$

Notes:

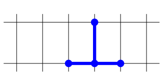
Stencil — CFL Condition



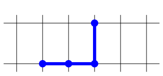
$$0 \le \frac{\lambda_p \Delta t}{\Delta x} \le 1, \quad \forall p$$



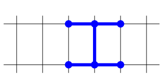
$$-1 \le \frac{\lambda_p \Delta t}{\Delta x} \le 0, \quad \forall p$$



$$-1 \le \frac{\lambda_p \Delta t}{\Delta x} \le 1, \quad \forall p$$



$$0 \le \frac{\lambda_p \Delta t}{\Delta x} \le 2, \quad \forall p$$



$$-\infty < \frac{\lambda_p \Delta t}{\Delta x} < \infty, \quad \forall p$$

Notes: