AMath 574 March 7, 2011

Today:

- Finite volume methods for nonlinear systems
- Wave propagation algorithms
- Approximate Riemann solvers

Wednesday:

- More about finite volume methods

Friday:

- Projects, What else??

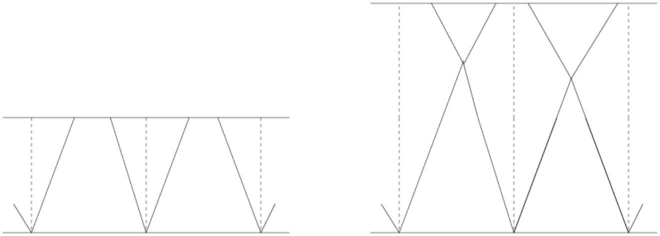
Reading: Chapter 15

Projects: Make an appointment this week, and see

http://www.clawpack.org/links/burgersadv

Godunov's method on a nonlinear system

Solve Riemann problems and average solution after time Δt.



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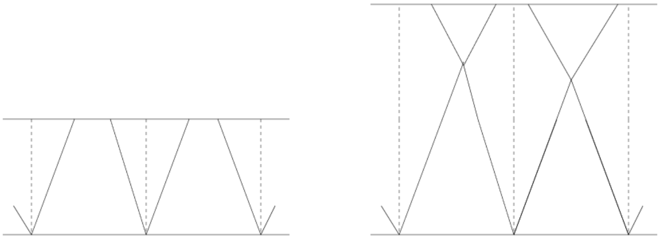

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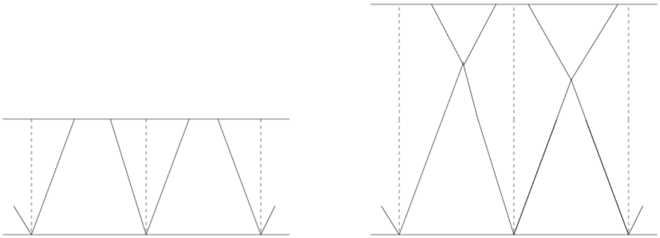

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We do not want to compute nonlinear interaction of waves!

But can compute averages from edge fluxes without doing so!

Or with wave-propagation algorithm...

Upwind wave-propagation algorithm

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\sum_{p=1}^{m}(\lambda^p)^+\mathcal{W}_{i-1/2}^p + \sum_{p=1}^{m}(\lambda^p)^-\mathcal{W}_{i+1/2}^p\right]$$

or

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right].$$

where the fluctuations are defined by

$$\mathcal{A}^-\Delta Q_{i-1/2} = \sum_{p=1}^{m}(\lambda^p)^-\mathcal{W}_{i-1/2}^p, \quad \text{left-going}$$

$$\mathcal{A}^+\Delta Q_{i-1/2} = \sum_{p=1}^{m}(\lambda^p)^+\mathcal{W}_{i-1/2}^p, \quad \text{right-going}$$

All shock solution to the nonlinear Riemann problem

For the wave-propagation algorithm we need jump discontinuities $\mathcal{W}^p_{i-1/2}$.

All-shock Riemann solution: Ignore rarefaction waves and use intersections of Hugoniot loci to define Riemann solution.

Correct solution in some cases.

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Transonic rarefactions can be handled by modifying $\mathcal{A}^{\pm}\Delta Q_{i-1/2}$, the flux-difference splitting used in 1st order terms.

Still use shock waves for high-resolution corrections.

Upwind wave-propagation algorithm

First order Godunov method:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right]$$

where

$$\mathcal{A}^-\Delta Q_{i-1/2} = \sum_{p=1}^{m} (s_{i-1/2}^p)^- \mathcal{W}_{i-1/2}^p,$$

$$\mathcal{A}^+\Delta Q_{i-1/2} = \sum_{p=1}^{m} (s_{i-1/2}^p)^+ \mathcal{W}_{i-1/2}^p,$$

May need to modify these by an entropy fix.

Entropy fix

Various approaches possible.

1. Compute "exact" value $q^{\Downarrow}(Q_{i-1}, Q_i)$ and set

$$\begin{aligned}\mathcal{A}^-\Delta Q_{i-1/2} &= f(q^{\Downarrow}) - f(Q_{i-1}),\\ \mathcal{A}^+\Delta Q_{i-1/2} &= f(Q_i) - f(q^{\Downarrow}).\end{aligned}$$

2. Split transonic wave $\mathcal{W}^p_{i-1/2}$ between $\mathcal{A}^-\Delta Q_{i-1/2}$ and $\mathcal{A}^+\Delta Q_{i-1/2}$.

Approximate Riemann solvers

For nonlinear problems, computing the exact solution to each Riemann problem may not be possible, or too expensive.

Often the nonlinear problem $q_t + f(q)_x = 0$ is approximated by

$$q_t + A_{i-1/2} q_x = 0, \qquad q_\ell = Q_{i-1}, \quad q_r = Q_i$$

for some choice of $A_{i-1/2} \approx f'(q)$ based on data $Q_{i-1},\ Q_i$.

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Solve linear system for $\alpha_{i-1/2}$: $Q_i - Q_{i-1} = \sum_p \alpha_{i-1/2}^p r_{i-1/2}^p$.

Waves $\mathcal{W}_{i-1/2}^p = \alpha_{i-1/2}^p r_{i-1/2}^p$ propagate with speeds $s_{i-1/2}^p$,

$r_{i-1/2}^p$ are eigenvectors of $A_{i-1/2}$,
$s_{i-1/2}^p$ are eigenvalues of $A_{i-1/2}$.

Approximate Riemann Solvers

Approximate true Riemann solution by set of waves consisting of finite jumps propagating at constant speeds.

Local linearization:

Replace $q_t + f(q)_x = 0$ by

$$q_t + \hat{A} q_x = 0,$$

where $\hat{A} = \hat{A}(q_l, q_r) \approx f'(q_{ave})$.

Then decompose

$$q_r - q_l = \alpha^1 \hat{r}^1 + \cdots \alpha^m \hat{r}^m$$

to obtain waves $\mathcal{W}^p = \alpha^p \hat{r}^p$ with speeds $s^p = \hat{\lambda}^p$.

Approximate Riemann solvers

$$q_t + \hat{A}_{i-1/2} q_x = 0, \qquad q_\ell = Q_{i-1}, \quad q_r = Q_i$$

Often $\hat{A}_{i-1/2} = f'(Q_{i-1/2})$ for some choice of $Q_{i-1/2}$.

In general $\hat{A}_{i-1/2} = \hat{A}(q_\ell,\ q_r)$.

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Roe conditions for consistency and conservation:

- $\hat{A}(q_\ell, q_r) \to f'(q^*)$ as $q_\ell, q_r \to q^*$,
- $\hat{A}$ diagonalizable with real eigenvalues,
- For conservation in wave-propagation form,

$$\hat{A}_{i-1/2}(Q_i - Q_{i-1}) = f(Q_i) - f(Q_{i-1}).$$

Roe Solver

Solve $q_t + \hat{A} q_x = 0$ where $\hat{A}$ satisfies

$$\hat{A}(q_r - q_l) = f(q_r) - f(q_l).$$

Then:

- Good approximation for weak waves (smooth flow)
- Single shock captured exactly:

  $f(q_r) - f(q_l) = s(q_r - q_l) \implies q_r - q_l$ is an eigenvector of $\hat{A}$

- Wave-propagation algorithm is conservative since

  $$\mathcal{A}^- \Delta Q_{i-1/2} + \mathcal{A}^+ \Delta Q_{i-1/2} = \sum s^p_{i-1/2} \mathcal{W}^p_{i-1/2} = A \sum \mathcal{W}^p_{i-1/2}.$$

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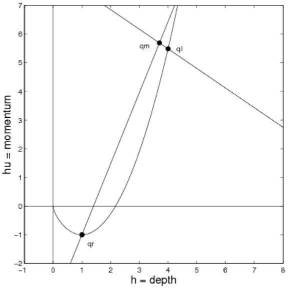
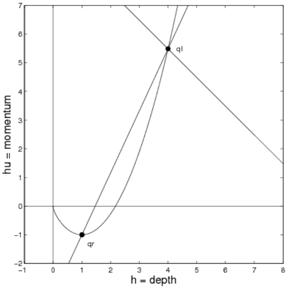
Roe average $\hat{A}$ can be determined analytically for many important nonlinear systems (e.g. Euler, shallow water).

Approximate solution to single wave

Suppose q_ℓ lies on some Hugoniot locus of q_r (and vice versa!):





$\hat{Q}_{i-1/2} = \frac{1}{2}(Q_{i-1} + Q_i)$ $\hat{Q}_{i-1/2} =$ Roe average

Straight lines are eigendirections of $f'(\hat{Q}_{i-1/2})$.

Approximate Riemann Solvers

How to use?

One approach: determine $Q^* =$ state along $x/t = 0$,

$$Q^* = Q_{i-1} + \sum_{p:s^p<0} \mathcal{W}^p, \quad F_{i-1/2} = f(Q^*),$$

$$\mathcal{A}^-\Delta Q = F_{i-1/2} - f(Q_{i-1}), \quad \mathcal{A}^+\Delta Q = f(Q_i) - F_{i-1/2}.$$

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Wave-propagation algorithm uses:

$$\mathcal{A}^-\Delta Q = \sum_{p:s^p<0} s^p\mathcal{W}^p, \qquad \mathcal{A}^+\Delta Q = \sum_{p:s^p>0} s^p\mathcal{W}^p.$$

Conservative only if $\mathcal{A}^-\Delta Q + \mathcal{A}^+\Delta Q = f(Q_i) - f(Q_{i-1})$.

This holds for Roe solver.

Approximate Riemann solvers

For a scalar problem, we can easily satisfy the Roe condition

$$\hat{A}_{i-1/2}(Q_i - Q_{i-1}) = f(Q_i) - f(Q_{i-1}).$$

by choosing

$$\hat{A}_{i-1/2} = \frac{f(Q_i) - f(Q_{i-1})}{Q_i - Q_{i-1}}.$$

Approximate Riemann solvers

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Then $r^1_{i-1/2} = 1$ and $s^1_{i-1/2} = \hat{A}_{i-1/2}$ (scalar!).

Note: This is the Rankine-Hugoniot shock speed.

$\Longrightarrow$ shock waves are correct,
rarefactions replaced by entropy-violating shocks.

Shallow water equations

$h(x,t) =$ depth
$u(x,t) =$ velocity (depth averaged, varies only with x)

Conservation of mass and momentum hu gives system of two equations.

mass flux $= hu$,
momentum flux $= (hu)u + p$ where $p =$ hydrostatic pressure

$$h_t + (hu)_x = 0$$
$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$

Jacobian matrix:

$$f'(q) = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix}, \qquad \lambda = u \pm \sqrt{gh}.$$

Roe solver for Shallow Water

Given $h_l,\ u_l,\ h_r,\ u_r$, define

$$\bar{h} = \frac{h_l + h_r}{2}, \quad \hat{u} = \frac{\sqrt{h_l}u_l + \sqrt{h_r}u_r}{\sqrt{h_l} + \sqrt{hr}}$$

Then

$$\hat{A} = \text{Jacobian matrix evaluated at this average state}$$

satisfies

$$A(q_r - q_l) = f(q_r) - f(q_l).$$

- Roe condition is satisfied,
- Isolated shock modeled well,
- Wave propagation algorithm is conservative,
- High resolution methods obtained using corrections with limited waves.

Roe solver for Shallow Water

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Eigenvalues of $\hat{A} = f'(\hat{q})$ are:

$$\hat{\lambda}^1 = \hat{u} - \hat{c}, \quad \hat{\lambda}^2 = \hat{u} + \hat{c}, \quad \hat{c} = \sqrt{g\bar{h}}.$$

Eigenvectors:

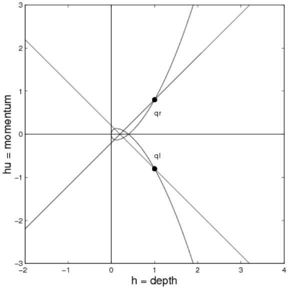
$$\hat{r}^1 = \begin{bmatrix} 1 \\ \hat{u} - \hat{c} \end{bmatrix}, \qquad \hat{r}^2 = \begin{bmatrix} 1 \\ \hat{u} + \hat{c} \end{bmatrix}.$$

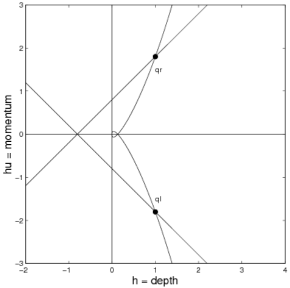
Examples in Clawpack 4.3 to be converted soon!

Potential failure of linearized solvers

Consider shallow water with $h_\ell = h_r$ and $u_r = -u_\ell \gg 1$.

Outflow away from interface $\implies$ small intermediate h_m.





With $u_r = 0.8$

Roe $h_m > 0$

With $u_r = 1.8$

Roe $h_m < 0$

HLL Solver

Harten – Lax – van Leer (1983): Use only 2 waves with

s^1 =minimum characteristic speed

s^2 =maximum characteristic speed

$$\mathcal{W}^1 = Q^* - Q_\ell, \qquad \mathcal{W}^2 = Q_r - Q^*$$

Conservation implies unique value for middle state Q^*:

$$s^1\mathcal{W}^1 + s^2\mathcal{W}^2 = f(Q_r) - f(Q_\ell)$$

$$\implies Q^* = \frac{f(Q_r) - f(Q_\ell) - s^2 Q_r + s^1 Q_\ell}{s^1 - s^2}.$$

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Choice of speeds:

- Max and min of expected speeds over entire problem,
- Max and min of eigenvalues of $f'(Q_\ell)$ and $f'(Q_r)$.

HLLE Solver

Einfeldt: Choice of speeds for gas dynamics (or shallow water) that guarantees positivity.

Based on characteristic speeds and Roe averages:

$$s^1_{i-1/2} = \min_p(\min(\lambda^p_i, \hat{\lambda}^p_{i-1/2})),$$
$$s^2_{i-1/2} = \max_p(\max(\lambda^p_{i+1}, \hat{\lambda}^p_{i-1/2})).$$

where

λ^p_i is the pth eigenvalue of the Jacobian $f'(Q_i)$,

$\hat{\lambda}^p_{i-1/2}$ is the pth eigenvalue using Roe average $f'(\hat{Q}_{i-1/2})$