AMath 574 February 23, 2011

Today:

- Nonlinear systems of conservation laws
- Shallow water equations
- Characteristics
- Rankine-Hugoniot condition, Hugoniot locus
- Solving Riemann problems

Friday:

- Integral curves, rarefaction waves

Reading: Chapter 13

Shallow water equations

$h(x,t) =$ depth
$u(x,t) =$ velocity (depth averaged, varies only with x)

Conservation of mass and momentum hu gives system of two equations.

mass flux $= hu$,
momentum flux $= (hu)u + p$ where $p =$ hydrostatic pressure

$$h_t + (hu)_x = 0$$
$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$

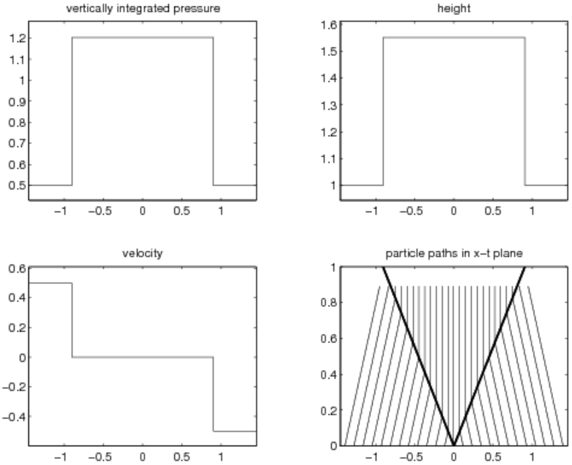
Jacobian matrix:

$$f'(q) = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix}, \qquad \lambda = u \pm \sqrt{gh}.$$

Two-shock Riemann solution for shallow water

Initially $h_l = h_r = 1, \quad u_l = -u_r = 0.5 > 0$

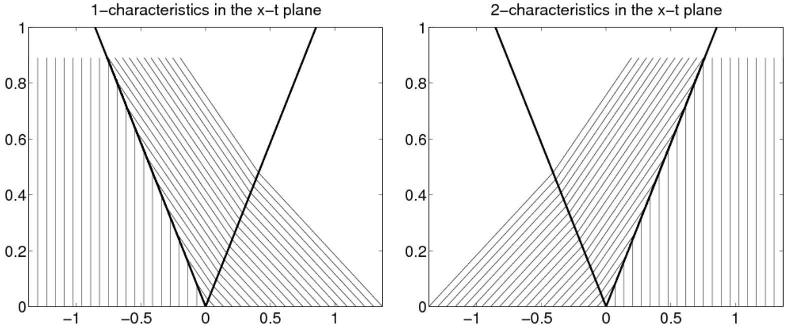
Solution at later time:



Two-shock Riemann solution for shallow water

Characteristic curves $X'(t) = u(X(t),t) \pm \sqrt{gh(X(t),t)}$

Slope of characteristic is constant in regions where q is constant. (Shown for $g = 1$ so $\sqrt{gh} = 1$ everywhere initially.)



Note that 1-characteristics impinge on 1-shock,
2-characteristics impinge on 2-shock.

An isolated shock

If an isolated shock with left and right states q_l and q_r is propagating at speed s

then the Rankine-Hugoniot condition must be satisfied:

$$f(q_r) - f(q_l) = s(q_r - q_l)$$

For a system $q \in \mathbb{R}^m$ this can only hold for certain pairs q_l, q_r:

For a linear system, $f(q_r) - f(q_l) = Aq_r - Aq_l = A(q_r - q_l)$.
So $q_r - q_l$ must be an eigenvector of $f'(q) = A$.

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$A \in \mathbb{R}^{m \times m} \implies$ there will be m rays through q_l in state space in the eigen-directions, and q_r must lie on one of these.

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For a nonlinear system, there will be m curves through q_l called the Hugoniot loci.

Hugoniot loci for shallow water

$$q = \begin{bmatrix} h \\ hu \end{bmatrix}, \qquad f(q) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}.$$

Fix $q_* = (h_*, u_*)$.

What states q can be connected to q_* by an isolated shock?

The Rankine-Hugoniot condition $s(q - q_*) = f(q) - f(q_*)$ gives:

$$\begin{aligned} s(h_* - h) &= h_* u_* - hu, \\ s(h_* u_* - hu) &= h_* u_*^2 - hu^2 + \frac{1}{2}g(h_*^2 - h^2). \end{aligned}$$

Two equations with 3 unknowns (h, u, s), so we expect 1-parameter families of solutions.

Hugoniot loci for shallow water

Rankine-Hugoniot conditions:

$$s(h_* - h) = h_* u_* - hu,$$
$$s(h_* u_* - hu) = h_* u_*^2 - hu^2 + \frac{1}{2} g (h_*^2 - h^2).$$

For any $h > 0$ we can solve for

$$u(h) = u_* \pm \sqrt{\frac{g}{2}\left(\frac{h_*}{h} - \frac{h}{h_*}\right)(h_* - h)}$$

$$s(h) = (h_* u_* - hu)/(h_* - h).$$

This gives 2 curves in h–hu space (one for $+$, one for $-$).

Hugoniot loci for shallow water

For any $h > 0$ we have a possible shock state. Set

$$h = h_* + \alpha,$$

so that $h = h_*$ at $\alpha = 0$, to obtain

$$hu = h_* u_* + \alpha \left[u_* \pm \sqrt{gh_* + \frac{1}{2} g\alpha(3 + \alpha/h_*)} \right].$$

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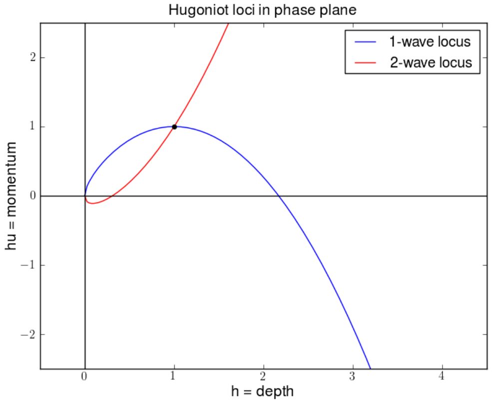
Hence we have

$$\begin{bmatrix} h \\ hu \end{bmatrix} = \begin{bmatrix} h_* \\ h_* u_* \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ u_* \pm \sqrt{g h_* + \mathcal{O}(\alpha)} \end{bmatrix} \qquad \text{as } \alpha \to 0.$$

Close to q_* the curves are tangent to eigenvectors of $f'(q_*)$
Expected since $f(q) - f(q_*) \approx f'(q_*)(q - q_*)$.

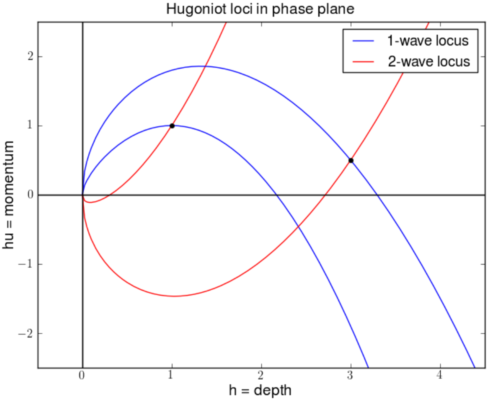
Hugoniot loci for one particular q_*

States that can be connected to q_* by a "shock"



Note: Might not satisfy entropy condition.

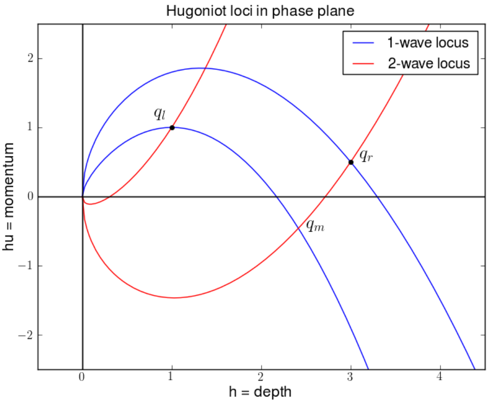
Hugoniot loci for two different states



“All-shock” Riemann solution:

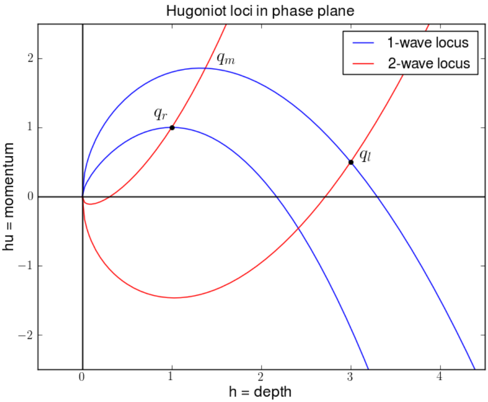
From q_l along 1-wave locus to q_m,
From q_r along 2-wave locus to q_m,

All-shock Riemann solution



From q_l along 1-wave locus to q_m,
From q_r along 2-wave locus to q_m,

All-shock Riemann solution



From q_l along 1-wave locus to q_m,
From q_r along 2-wave locus to q_m,

2-shock Riemann solution for shallow water

Given arbitrary states q_l and q_r, we can solve the Riemann problem with two shocks.

Choose q_m so that q_m is on the 1-Hugoniot locus of q_l and also q_m is on the 2-Hugoniot locus of q_r.

This requires

$$u_m = u_r + (h_m - h_r)\sqrt{\frac{g}{2}\left(\frac{1}{h_m} + \frac{1}{h_r}\right)}$$

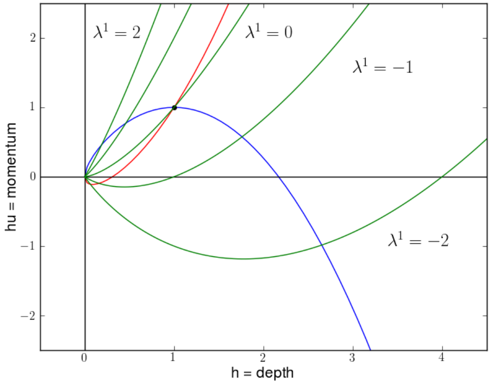
and

$$u_m = u_l - (h_m - h_l)\sqrt{\frac{g}{2}\left(\frac{1}{h_m} + \frac{1}{h_l}\right)}.$$

Equate and solve single nonlinear equation for h_m.

Hugoniot loci for one particular q_*

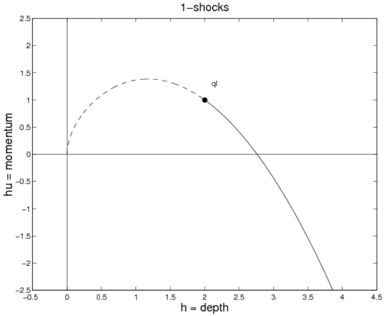
Green curves are contours of λ^1



Note: Increases in one direction only along blue curve.

Hugoniot locus for shallow water

States that can be connected to the given state by a 1-wave or 2-wave satisfying the R-H conditions:

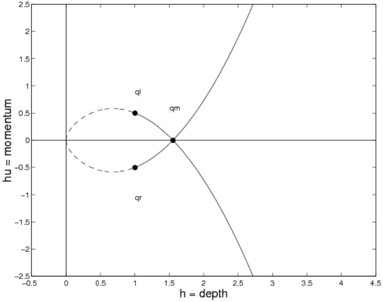


Solid portion: states that can be connected by shock satisfying entropy condition.

Dashed portion: states that can be connected with R-H condition satisfied but not the physically correct solution.

2-shock Riemann solution for shallow water

Colliding with $u_l = -u_r > 0$:



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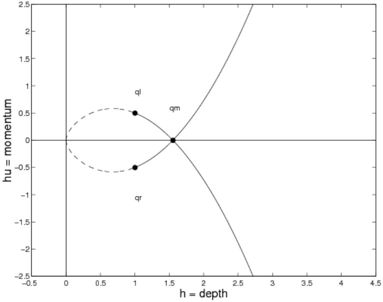


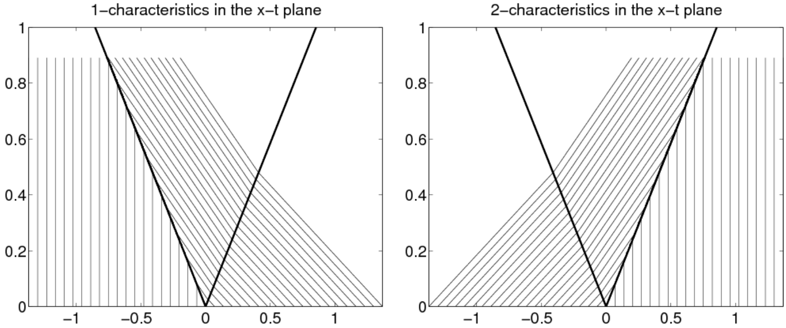
Entropy condition: Characteristics should impinge on shock:
λ^1 should decrease going from q_l to q_m,
λ^2 should increase going from q_r to q_m,

This is satisfied along solid portions of Hugoniot loci above, not satisfied on dashed portions (entropy-violating shocks).

Two-shock Riemann solution for shallow water

Characteristic curves $X'(t) = u(X(t), t) \pm \sqrt{gh(X(t), t)}$

Slope of characteristic is constant in regions where q is constant. (Shown for $g = 1$ so $\sqrt{gh} = 1$ everywhere initially.)



Note that 1-characteristics impinge on 1-shock,
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2-shock Riemann solution for shallow water

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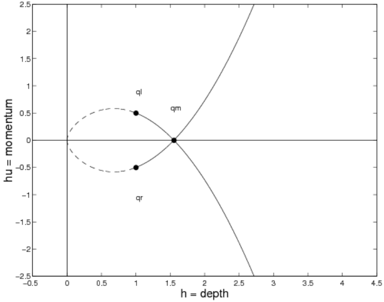
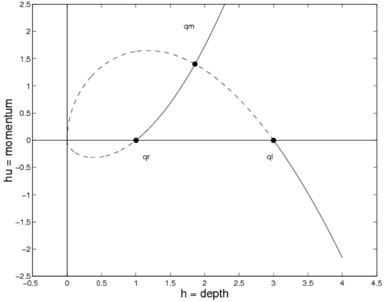


Dam break:



2-shock Riemann solution for shallow water

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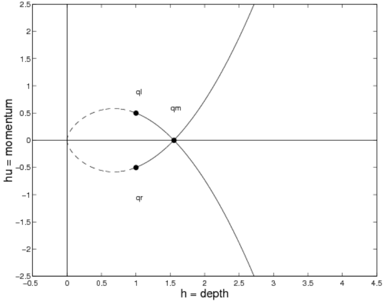

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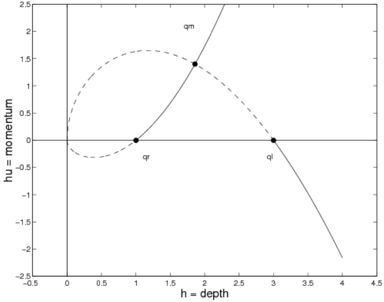

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Entropy-violatiing Riemann solution for dam break

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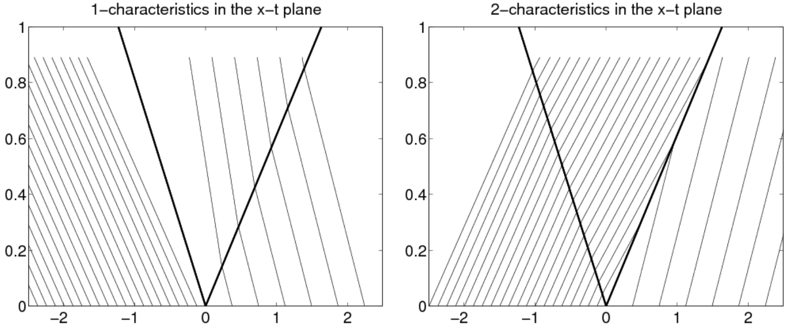


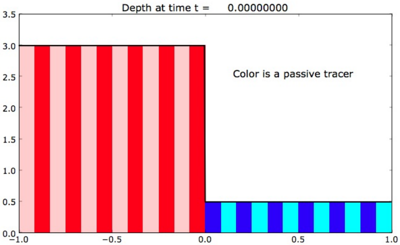
Note that 1-characteristics do not impinge on 1-shock, 2-characteristics impinge on 2-shock.

The Riemann problem

Dam break problem for shallow water equations

$$h_t + (hu)_x = 0$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$



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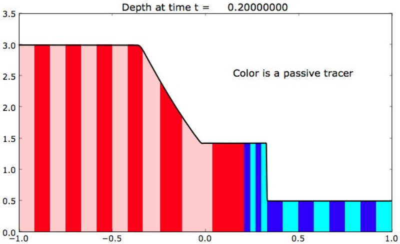

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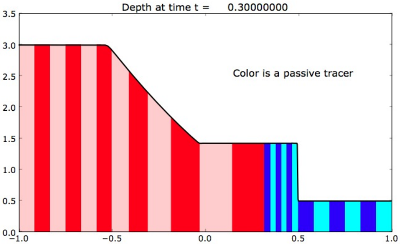

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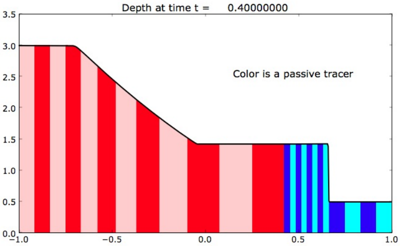

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