**AMath 574 February 2, 2011**

Today:

- Multi-dimensional advection
- Finite volume methods
- Dimensional splitting and fractional steps

Friday:

- Multi-dimensional wave propagation

Reading: Chapters 18, 19, 20



## Notes:



## First order hyperbolic PDE in 2 space dimensions

Advection equation: $q_t + uq_x + vq_y = 0$

First-order system: $q_t + Aq_x + Bq_y = 0$

where $q \in \mathbb{R}^m$ and $A, B \in \mathbb{R}^{m\times m}$.

Hyperbolic if $\cos(\theta)A + \sin(\theta)B$ is diagonalizable with real eigenvalues, for all angles $\theta$.

This is required so that plane-wave data gives a 1d hyperbolic problem:

$$q(x, y, 0) = \breve{q}(x\cos\theta + y\sin\theta)$$ (\breve q)

implies contours of $q$ in $x$–$y$ plane are orthogonal to $\theta$–direction.



## Notes:



## Plane wave solutions

Suppose

$$\begin{aligned} q(x, y, t) &= \breve{q}(x\cos\theta + y\sin\theta, t) \\ &= \breve{q}(\xi, t). \end{aligned}$$

Then:

$$\begin{aligned} q_x(x, y, t) &= \cos\theta\, \breve{q}_\xi(\xi, t) \\ q_y(x, y, t) &= \sin\theta\, \breve{q}_\xi(\xi, t) \end{aligned}$$

so

$$q_t + Aq_x + Bq_y = \breve{q}_t + (A\cos\theta + B\sin\theta)\breve{q}_\xi$$

and the 2d problem reduces to the 1d hyperbolic equation

$$\breve{q}_t(\xi, t) + (A\cos\theta + B\sin\theta)\breve{q}_\xi(\xi, t) = 0.$$



## Notes:

## Advection in 2 dimensions

Constant coefficient: $q_t + uq_x + vq_y = 0$

In this case solution for arbitrary initial data is easy:

$$q(x, y, t) = q(x - ut,\ y - vt,\ 0).$$

Data simply shifts at constant velocity $(u, v)$ in $x$-$y$ plane.

Variable coefficient:

Conservation form: $q_t + (u(x, y, t)q)_x + (v(x, y, t)q)_y = 0$

Advective form (color eqn): $q_t + u(x, y, t)q_x + v(x, y, t)q_y = 0$

Equivalent only if flow is divergence-free (incompressible):

$$\nabla \cdot \vec{u} = u_x(x, y, t) + v_y(x, y, t) = 0 \qquad \forall t \geq 0.$$



## Notes:



## Advection in 2 dimensions: characteristics

The characteristic curve $(X(t), Y(t))$ starting at some $(x_0, y_0)$ is determined by solving the ODEs

$$X'(t) = u(X(t),\ Y(t),\ t), \qquad X(0) = x_0$$
$$Y'(t) = v(X(t),\ Y(t),\ t), \qquad Y(0) = y_0.$$

How does $q$ vary along this curve?

$$\frac{d}{dt}q(X(t),\ Y(t),\ t) = X'(t)q_x(\cdots) + Y'(t)q_y(\cdots) + q_t(\cdots)$$

For color equation: $q_t + u(x, y, t)q_x + v(x, y, t)q_y = 0$

$q$ is constant along characterisitic (color is advected).



## Notes:



## Advection in 2 dimensions: characteristics

For conservative equation: $q_t + (u(x, y, t)q)_x + (v(x, y, t)q)_y = 0$

Can rewrite as $q_t + u(x, y, t)q_x + v(x, y, t)q_y = (u_x + v_y)q$

Along characteristic $q$ varies because of source term:

$$\frac{d}{dt}q(X(t),\ Y(t),\ t) = X'(t)q_x(\cdots) + Y'(t)q_y(\cdots) + q_t(\cdots)$$
$$= (\nabla \cdot \vec{u})q.$$

Conservative form models density of conserved quantity.

Mass in region advecting with the flow varies stays constant
but density increases if volume of region decreases,
or density decreases if volume of region increases.



## Notes:

## Acoustics in 2 dimensions

$$p_t + K_0(u_x + v_y) = 0$$
$$\rho_0 u_t + p_x = 0$$
$$\rho_0 v_t + p_y = 0$$

Note: pressure responds to compression or expansion and so $p_t$ is proportional to divergence of velocity.

Second and third equations are $F = ma$.

Gives hyperbolic system $q_t + Aq_x + Bq_y = 0$ with

$$q = \begin{bmatrix} p \\ u \\ v \end{bmatrix}, \quad A = \begin{bmatrix} 0 & K_0 & 0 \\ 1/\rho_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & K_0 \\ 0 & 0 & 0 \\ 1/\rho_0 & 0 & 0 \end{bmatrix}.$$



## Notes:



## Acoustics in 2 dimensions

$$q = \begin{bmatrix} p \\ u \\ v \end{bmatrix}, \quad A = \begin{bmatrix} 0 & K_0 & 0 \\ 1/\rho_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & K_0 \\ 0 & 0 & 0 \\ 1/\rho_0 & 0 & 0 \end{bmatrix}.$$

Plane waves:

$$A\cos\theta + B\sin\theta = \begin{bmatrix} 0 & K_0\cos\theta & K_0\sin\theta \\ \cos\theta/\rho_0 & 0 & 0 \\ \sin\theta/\rho_0 & 0 & 0 \end{bmatrix}.$$

Eigenvalues: $\lambda^1 = -c_0, \quad \lambda^2 = 0, \quad \lambda^3 = +c_0$ where $c_0 = \sqrt{K_0/\rho_0}$

Independent of angle $\theta$.

Isotropic: sound propagates at same speed in any direction.

Note: Zero wave speed for "shear wave" with variation only in velocity in direction $(-\sin\theta,\ \cos\theta)$. (Fig 18.1)



## Notes:



## Diagonalization 2 dimensions

Can we diagonalize system $q_t + Aq_x + Bq_y = 0$?

Only if $A$ and $B$ have the same eigenvectors!

If $A = R\Lambda R^{-1}$ and $B = RMR^{-1}$, then let $w = R^{-1}q$ and

$$w_t + \Lambda w_x + M w_y = 0$$

This decouples into scalar advection equations for each component of $w$:

$$w_t^p + \lambda^p w_x^p + \mu^p w_y^p = 0 \implies w^p(x,y,t) = w^p(x - \lambda^p t,\ y - \mu^p t,\ 0).$$

Note: In this case information propagates only in a finite number of directions $(\lambda^p,\ \mu^p)$ for $p = 1,\ \ldots,\ m$.

This is not true for most coupled systems, e.g. acoustics.



## Notes:

## Acoustics in 2 dimensions

$$p_t + K_0(u_x + v_y) = 0$$
$$\rho_0 u_t + p_x = 0$$
$$\rho_0 v_t + p_y = 0$$

$$A = \begin{bmatrix} 0 & K_0 & 0 \\ 1/\rho_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad R^x = \begin{bmatrix} -Z_0 & 0 & Z_0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Solving $q_t + Aq_x = 0$ gives pressure waves in $(p, u)$.
$x$-variations in $v$ are stationary.

$$B = \begin{bmatrix} 0 & 0 & K_0 \\ 0 & 0 & 0 \\ 1/\rho_0 & 0 & 0 \end{bmatrix} \qquad R^y = \begin{bmatrix} -Z_0 & 0 & Z_0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Solving $q_t + Bq_y = 0$ gives pressure waves in $(p, v)$.
$y$-variations in $u$ are stationary.



## Notes:



## Dimensional Splitting

Hyperbolic system in 2d: $q_t + Aq_x + Bq_y = 0$
$x$-sweeps : $q_t + Aq_x = 0$
$y$-sweeps : $q_t + Bq_y = 0$.

Use one-dimensional high-resolution methods for each,
"Godunov splitting" if clawdata.order_trans $= -1$,
"Strang splitting" if clawdata.order_trans $= -2$,

- Easy to extend good one-dimensional methods to 2D or 3D.
- Often very effective and efficient.
- May suffer from lack of isotropy.
- May be hard to use with AMR, complex geometry.

Alternative: Unsplit method if clawdata.order_trans $\geq 0$.



## Notes:



## Fractional step method for a linear PDE

$q_t = (\mathcal{A} + \mathcal{B})q$ dimensional splitting: $\mathcal{A} = A\partial_x$, $\mathcal{B} = B\partial_y$.

Then

$$q_{tt} = (\mathcal{A} + \mathcal{B})q_t = (\mathcal{A} + \mathcal{B})^2 q,$$

and so

$$\begin{aligned} q(x, \Delta t) &= q(x, 0) + \Delta t(\mathcal{A} + \mathcal{B})q(x, 0) + \frac{1}{2}\Delta t^2(\mathcal{A} + \mathcal{B})^2 q(x, 0) + \cdots \\ &= \left(I + \Delta t(\mathcal{A} + \mathcal{B}) + \frac{1}{2}\Delta t^2(\mathcal{A} + \mathcal{B})^2 + \cdots\right) q(x, 0) \end{aligned}$$

Solution operator: $q(x, \Delta t) = e^{\Delta t(\mathcal{A}+\mathcal{B})}q(x, 0)$.

With the fractional step method, we instead compute

$$q^*(x, \Delta t) = e^{\Delta t \mathcal{A}} q(x, 0),$$

and then

$$q^{**}(x, \Delta t) = e^{\Delta t \mathcal{B}} e^{\Delta t \mathcal{A}} q(x, 0),$$



## Notes:

## Splitting error

$$q(x,\Delta t) - q^{**}(x,\Delta t) = \left(e^{\Delta t(\mathcal{A}+\mathcal{B})} - e^{\Delta t\mathcal{B}}e^{\Delta t\mathcal{A}}\right) q(x,0)$$

Combining 2 steps gives:

$$\begin{aligned} q^{**}(x,\Delta t) &= \left(I + \Delta t\mathcal{B} + \frac{1}{2}\Delta t^2\mathcal{B}^2 + \cdots\right)\left(I + \Delta t\mathcal{A} + \frac{1}{2}\Delta t^2\mathcal{A}^2 + \cdots\right) q(x,0) \\ &= \left(I + \Delta t(\mathcal{A}+\mathcal{B}) + \frac{1}{2}\Delta t^2(\mathcal{A}^2 + 2\mathcal{B}\mathcal{A} + \mathcal{B}^2) + \cdots\right) q(x,0). \end{aligned}$$

In true solution operator,

$$\begin{aligned} (\mathcal{A}+\mathcal{B})^2 &= (\mathcal{A}+\mathcal{B})(\mathcal{A}+\mathcal{B}) \\ &= \mathcal{A}^2 + \mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A} + \mathcal{B}^2. \end{aligned}$$



## Notes:



## Splitting error

$$\begin{aligned} q(x,\Delta t) - q^{**}(x,\Delta t) &= \left(e^{\Delta t(\mathcal{A}+\mathcal{B})} - e^{\Delta t\mathcal{B}}e^{\Delta t\mathcal{A}}\right) q(x,0) \\ &= \frac{1}{2}\Delta t^2(\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A})q(x,0) + O(\Delta t^3). \end{aligned}$$

There is a splitting error unless the two operators commute.

No splitting error for constant coefficient advection:

$$\mathcal{A} = u\partial_x, \quad \mathcal{B} = v\partial_y \quad \mathcal{A}\mathcal{B}q = \mathcal{B}\mathcal{A}q = uvq_{xy}$$

There is a splitting error if $u, v$ are varying:

$$\begin{aligned} \mathcal{A}\mathcal{B}q &= u(x,y)\partial_x(v(x,y)\partial_y)q = uvq_{xy} + uv_xq_y, \\ \mathcal{B}\mathcal{A}q &= v(x,y)\partial_y(u(x,y)\partial_x)q = uvq_{xy} + vu_yq_x. \end{aligned}$$

There is a splitting error for acoustics since $ABq_{xy} \neq BAq_{xy}$.



## Notes:



## Strang splitting

- Time step $\Delta t/2$ on A-problem,
- Time step $\Delta t$ on B-problem,
- Time step $\Delta t/2$ on A-problem.

Formally second order if each solution method is.

$$\left(e^{\Delta t(\mathcal{A}+\mathcal{B})} - e^{\frac{1}{2}\Delta t\mathcal{A}}e^{\Delta t\mathcal{B}}e^{\frac{1}{2}\Delta t\mathcal{A}}\right) q(x,0) = O(\Delta t^3).$$

In practice often little difference from “first order Godunov splitting”



## Notes:

## Example of splitting error for source term

Advection + decay: $q_t + uq_x = -\lambda(x)q$

Take $\mathcal{A} = -u\partial_x$ and $\mathcal{B} = \lambda(x)\partial_x$.

Then:

$$\begin{aligned}\mathcal{AB}q &= -u\partial_x(\lambda(x)q_x) = -u\lambda(x)q_{xx} - u\lambda'(x)q_x \\ \mathcal{BA}q &= -\lambda(x)uq_{xx}\end{aligned}$$

Splitting error unless $\lambda(x) =$ constant

Source term in Clawpack: Provide `src1.f` in 1d
or `src2.f` in 2d that advances $Q$ in each cell by time $\Delta t$.

Set `clawdata.src_split = 1` (or = `2` for Strang splitting)



## Notes:



## Wave propagation algorithms in 2D

Clawpack requires:

Normal Riemann solver `rpn2.f`
Solves 1d Riemann problem $q_t + Aq_x = 0$
Decomposes $\Delta Q = Q_{ij} - Q_{i-1,j}$ into $\mathcal{A}^+\Delta Q$ and $\mathcal{A}^-\Delta Q$.
For $q_t + Aq_x + Bq_y = 0$, split using eigenvalues, vectors:

$$A = R\Lambda R^{-1} \implies A^- = R\Lambda^- R^{-1}, A^+ = R\Lambda^+ R^{-1}$$

Input parameter `ixy` determines if it's in $x$ or $y$ direction.
In latter case splitting is done using $B$ instead of $A$.
This is all that's required for dimensional splitting.

Transverse Riemann solver `rpt2.f`
Decomposes $\mathcal{A}^+\Delta Q$ into $\mathcal{B}^-\mathcal{A}^+\Delta Q$ and $\mathcal{B}^+\mathcal{A}^+\Delta Q$ by splitting this vector into eigenvectors of $B$.

(Or splits vector into eigenvectors of $A$ if `ixy=2`.)



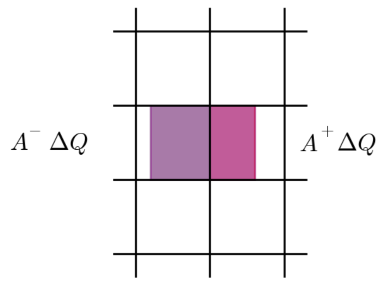
## Notes:



## Wave propagation algorithm for $q_t + Aq_x + Bq_y = 0$

Decompose $A = A^+ + A^-$ and $B = B^+ + B^-$.

For $\Delta Q = Q_{ij} - Q_{i-1,j}$:





## Notes: