Review of Interpolation

1 General interpolation problem

Given a set of discrete points x_i for $i = 1,\ 2,\ \ldots,\ n$ and function values F_i, determine a function $\phi(x)$ passing through these points,

$$\phi(x_i) = F_i \quad \text{for } i = 1,\ 2,\ \ldots,\ n. \tag{1}$$

We use the notation $\text{Int}(x_1, \ldots, x_n)$ to denote the smallest interval containing all these points (which need not be in increasing order but which are assumed to be distinct).

Some uses of interpolation:

- May only have discrete data values and want to estimate values in between, $x \in \text{Int}(x_1, \ldots, x_n)$. This is the origin of the term *interpolation*. We might also use this function to *extrapolate* if we evaluate it outside the interval where data is given.
- May know true function $F(x)$ but want to approximate it by a function $\phi(x)$ that is cheaper to evaluate, or easier to work with symbolically (to differentiate or integrate, for example).
- As a starting point for deriving numerical methods for differential equations (or for integral equations or numerical integration).

There are infinitely many possible functions ϕ. Typically ϕ is chosen to be a linear combination of some n given *basis functions* $\phi_1(x),\ \ldots,\ \phi_n(x)$,

$$\phi(x) = c_1\phi_1(x) + \cdots + c_n\phi_n(x). \tag{2}$$

Then condition (1) gives a linear system of n equations to solve for the coefficients $c_1,\ \ldots,\ c_n$,

$$\begin{bmatrix} \phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_n(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \cdots & \phi_n(x_2) \\ \vdots & & & \vdots \\ \phi_1(x_n) & \phi_2(x_n) & \cdots & \phi_n(x_n) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix} \tag{3}$$

This system can we written as $\Phi c = F$. Different choices of basis functions lead to different types of interpolation. Using trigonometric functions gives Fourier series, for example.

2 Polynomial interpolation

Through any n points there is a unique interpolating polynomial $p(x)$ of degree $n-1$. There are many ways to represent this function depending on what basis is chosen for $\mathcal{P}_{n-1}$, the set of all polynomials of degree $n-1$.

2.1 Monomial basis

$$\phi_1(x) = 1, \quad \phi_2(x) = x, \quad \phi_3(x) = x^2, \quad \ldots, \quad \phi_n(x) = x^{n-1}. \tag{4}$$

The matrix Φ appearing in (3) is then the *Vandermonde matrix*. This matrix may be quite ill-conditioned.

2.2 Lagrange basis

$$\phi_j(x) = \prod_{\substack{i=1 \\ i\neq j}}^{n} \frac{(x - x_i)}{(x_j - x_i)}. \tag{5}$$

This is a polynomial of degree $n - 1$. Note that

$$\phi_j(x_i) = \delta_{ij} = \begin{cases} 1 & \text{if} i = j \\ 0 & \text{if} i \neq j. \end{cases}$$

Then the matrix in (3) is the identity matrix and $c_i = F_i$. The coefficients are easy to determine in this form but the basis functions are a bit cumbersome.

2.3 Newton form

The Newton form of the interpolating polynomial is

$$p(x) = c_1 + c_2(x - x_1) + c_3(x - x_1)(x - x_2) + \cdots + c_n(x - x_1)(x - x_2)\cdots(x - x_{n-1}). \tag{6}$$

For these basis functions the matrix Φ is *lower triangular* and the c_i may be found by forward substitution. Alternatively they are most easily computed using *divided differences*, $c_i = F[x_1, \ldots, x_i]$. These can be computed from a tableau of the form

$$\begin{array}{lllll} x_1 & F[x_1] & & & \\ & & F[x_1, x_2] & & \\ x_2 & F[x_2] & & F[x_1, x_2, x_3] & \\ & & F[x_2, x_3] & & \\ x_3 & F[x_3] & & & \end{array} \tag{7}$$

where

$$F[x_j] = F_j$$

and for $k > 0$,

$$F[x_j, \ldots, x_{j+k}] = \frac{F[x_{j+1}, \ldots, x_{j+k}] - F[x_j, \ldots, x_{j+k-1}]}{x_{j+k} - x_j}. \tag{8}$$

Then the Newton form can be built up as follows:

$p_0(x) = F[x_1]$

is the polynomial of degree 0 interpolating at x_1

$p_1(x) = F[x_1] + F[x_1, x_2](x - x_1)$

is the polynomial of degree 1 interpolating at x_1, x_2

$p_1(x) = F[x_1] + F[x_1, x_2](x - x_1) + F[x_1, x_2, x_3](x - x_1)(x - x_2)$

is the polynomial of degree 2 interpolating at x_1, x_2, x_3

etc.

Each step we add a term which vanishes at all the preceding interpolation points and makes the function also interpolate at one new point. Note that the coefficients of previous basis functions do not change.

Relation to Taylor series. Note that

$$F[x_j, x_{j+1}] = \frac{F_{j+1} - F_j}{x_{j+1} - x_j}$$

approximates a derivative $F'(x_j)$. Similarly, if $x_j, \dots, x_{j+k}$ are close together then

$$F[x_j, \dots, x_{j+k}] \approx \frac{1}{k!} F^{(k)}(x_j) \tag{9}$$

where $F^{(k)}(x)$ is the kth derivative. In fact one can show that

$$F[x_j, \dots, x_{j+k}] = \frac{1}{k!} F^{(k)}(\xi) \tag{10}$$

for some ξ lying in the interval $\text{Int}(x_j, \dots, x_{j+k})$. The Newton form (6) thus is similar to the Taylor series

$$F(x) = F(x_1) + F'(x_1)(x - x_1) + \frac{1}{2!} F''(x_1)(x - x_1)^2 + \cdots \tag{11}$$

and gives this in the limit as $x_j \to x_1$ for all j.

2.4 Error in polynomial interpolation

Suppose $F(x)$ is a smooth function, we evaluate $F_i = F(x_i)$ $(i = 1,\ 2,\ \dots,\ n)$, and now fit a polynomial $p(x)$ of degree $n - 1$ through these points. How well does $p(\bar{x})$ approximate $F(\bar{x})$ at some other point $\bar{x}$?

Note that we could add $\bar{x}$ as another interpolation point and create an interpolating polynomial $\bar{p}(x)$ of degree n that interpolates also at this point,

$$\bar{p}(x) = p(x) + F[x_1,\ \dots,\ x_n,\ \bar{x}](x - x_1) \cdots (x - x_n).$$

Then $\bar{p}(\bar{x}) = F(\bar{x})$ and so

$$F(\bar{x}) - p(\bar{x}) = F[x_1,\ \dots,\ x_n,\ \bar{x}](\bar{x} - x_1) \cdots (\bar{x} - x_n).$$

Using (10), we obtain an error formula similar to the remainder formula for Taylor series, which states that if $p(x)$ is given by (6), then

$$F(x) - p(x) = \frac{1}{n!} F^{(n)}(\xi)(x - x_1) \cdots (x - x_n) \tag{12}$$

where ξ is some point lying in $\text{Int}(x, x_1, \dots, x_n)$. How large this is depends on

- How close the point x is to the interpolation points $x_1, \dots, x_n$,
- How small the derivative $F^{(n)}(\xi)$ is over this interval, *i.e.*, how smooth the function is.

For a given x we don't know exactly what ξ is in general, but we can often use this to obtain an *error bound* of the form

$$|p(x) - F(x)| \le K|(x - x_1) \cdots (x - x_n)| \tag{13}$$

where $K = \frac{1}{n!} \max_{\xi \in \text{Int}(x_1, \dots, x_n)} |F^{(n)}(\xi)|$.

2.5 Chebyshev points

Suppose we wish to approximate some $F(x)$ over $[-1,1]$ by a polynomial of degree $n-1$ based on interpolation at some n points in this interval. If we are free to pick these points however we want, then we might want to minimize

$$\max_{-1\leq x\leq 1} |(x-x_1)\cdots(x-x_n)|. \tag{14}$$

This mini-max problem can be solved and the best points are the n *Chebyshev points*

$$x_j = \cos\theta_j \qquad \text{where} \quad \theta_j = \frac{(j-1/2)\pi}{n} \tag{15}$$

for $j = 1, 2, \ldots, n$. Now let

$$T_n(x) = (x-x_1)\cdots(x-x_n) \tag{16}$$

be the polynomial of degree n with these points as its roots (which appears in (12)). This is the *Chebyshev polynomial* of degree n. The first few are

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^3 - 3x \end{aligned}$$

In general they satisfy a 3-term recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \tag{17}$$

For $-1 \leq x \leq 1$ they also have the property that

$$T_n(x) = \cos(n \arccos x). \tag{18}$$

This doesn't look so much like a polynomial in this form, but shows that $T_n(x)$ oscillates between ± 1 over this interval.