AMath 584
Homework 2
Due to dropbox by 6pm PDT, October 21, 2011

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Submit a pdf file to the dropbox. Use latex if possible!!

Problem #1.

Let

$$A = \begin{bmatrix} 4 & 0 \\ 3 & 0 \\ 0 & 0 \end{bmatrix}.$$

(a) Determine the 2-norm of A by finding a vector x that maximizes $\|Ax\|/\|x\|$. You can use Matlab/Python to check your answer, but you should be able to figure it out by considering the form of Ax. Explain your answer.

(b) Using the idea from the proof of Theorem 4.1, determine the SVD of A by hand, showing your steps. Again you might want to check it, but remember that the SVD is not entirely unique.

(c) Using the idea from the proof of Theorem 4.1, determine the SVD of

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solution:

Problem #2.

Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2+m_3)\times(n_1+n_2)}$$

be a block matrix, with each A_{ij} representing an $m_i \times n_j$ matrix, for some positive integers m_1, m_2, m_3, n_1, n_2.

(a) Find block matrices M_1 and M_2 so that

$$A_{31} = M_1 A M_2.$$

You can write M_1 and M_2 in terms of the 0 matrix and the I matrix, but make it clear what size each is, e.g., $I_{n_1\times n_1}$ would be an $n_1 \times n_1$ identity matrix.

(b) Let $\|\cdot\|$ be any vector norm and the associated matrix norm. Use the decomposition above to show that $\|A_{31}\| \leq \|A\|$. (In fact by similar reasoning the norm of any block of a matrix is bounded by the norm of the full matrix.)

Solution:

Problem #3.

Let $u,\ v \in \mathbb{C}^m$ and consider the matrix $A = I + uv^* \in \mathbb{C}^{m\times m}$.

(a) Show that if A is invertible then $A^{-1} = I + \alpha uv^*$ for some scalar α and determine this value.

(b) What condition is required on u and v so that A is invertible?

Solution:

Problem #4. Let $u \in \mathbb{C}^m$ be a unit vector, $\|u\| = 1$ (2-norm). The $m \times m$ identity matrix has a singular value decomposition of the form

$$I = [u\ U_2] \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} u^* \\ U_2^* \end{bmatrix}$$

where the columns of $U_2 \in \mathbb{C}^{m\times(m-1)}$ form an orthonormal basis for the orthogonal complement of $\text{span}(u) = \langle u \rangle$, which is the space of all vectors orthogonal to u.

(a) Using this fact, determine a singular value decomposition of the matrix $I + \beta uu^*$ where β is an arbitrary complex scalar. Remember that the singular values must be real and non-negative!

(b) What is the 2-norm of $A = I + \beta uu^*$? Determine a vector x so that $\|Ax\| = \|A\|\|x\|$.

Solution:

Problem #5.

(a) Exercise 5.4 in the book: Suppose $A \in \mathbb{C}^{m\times m}$ has an SVD $A = U\Sigma V^*$. Find an eigenvalue decomposition of the form $B = X\Lambda X^{-1}$ for the $2m \times 2m$ hermitian matrix

$$B = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}.$$

Hints: The eigenvector matrix X will be a $2m \times 2m$ matrix that can be written in block form using the U and V matrices. The Λ matrix will be diagonal and the $2m$ eigenvalues are $\pm\sigma_j$ for each singular value of A.

(b) Since the matrix B is hermitian, the matrix of eigenvectors X should be unitary. Check that this is so.

Solution:

Problem #6.

We've talked about the SVD of an $m \times n$ matrix when $m > n$ and the figure on p. 28 shows the full SVD and how it relates to the reduced SVD. Figure out the corresponding form for the case $m < n$. You don't need to try to draw a figure, but make sure you understand it, and in particular write out the SVD for the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

You can use Matlab or Python to help, and recall that $1/\sqrt{2} \approx 0.7071$.

Solution:

Problem #7.

(a) Do exercise 6.2 in the book. Show the matrix E in the cases $m = 4$ and $m = 5$.

(b) Determine the SVD of E in the case $m = 5$. (Use Matlab/Python if you wish to help figure this out.)

(c) Based on the SVD determined in (b), what is an orthonormal basis for the range of E? Explain why this makes sense in relation to the effect that E has on a vector.

Solution: