AMath 584
Homework 1
Due to dropbox by 6pm PDT, October 7, 2011

Name: INSERT YOUR NAME HERE
UWNetID: INSERT YOUR NETID

Modify the latex file found on the webpage for the assignment in order to write up your solutions. Submit the resulting `pdf` file (again named `am584hw1.pdf`) to the dropbox. See the assignment page for more information.

If you can't get latex working for this assignment, you are welcome to use another approach, but please submit at `pdf` file to the dropbox. You can submit a combination of latex for some problems and another approach for others if you want — it would be good to start getting some experience with latex!

Problem #1.

If $x \in \mathbb{R}^2$ is interpreted as a point in the plane, then multiplication by the matrix

$$Q(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

rotates this point about the origin through an angle θ.

(a) Show that $Q(\theta)$ is unitary for any θ.

(b) Determine the inverse $Q(\theta)^{-1}$ and confirm that this matrix rotates through angle $-\theta$.

(c) Show that $Q(\theta_1)Q(\theta_2) = Q(\theta_1 + \theta_2)$ for any $\theta_1,\ \theta_2$.

Solution:

Problem #2.

Let $x \in \mathbb{R}^2$ with $x_2 \neq 0$, and let $v = \|x\|e_1 - x$ where $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\|\cdot\|$ is the 2-norm (here and in other problems if not specified otherwise).

Let

$$F = I - \left(\frac{2}{v^*v}\right) vv^*$$

as in equation (10.4) of Trefethen & Bau.

(a) Show that F is symmetric.

(b) Deleted!

(c) Show that F is orthogonal (unitary).

(d) Show that $Fx = \begin{bmatrix} \|x\| \\ 0 \end{bmatrix}$, so that multiplication by F introduces a 0 into the vector x.

(e) Show that multiplication of a general vector $y \in \mathbb{R}^2$ by F reflects the point y across a line in the plane (see Figure 10.2 in the book). What is the equation of this line?

Solution:

Problem #3.

An $m \times m$ *permutation matrix* P is obtained by permuting the columns of the $m \times m$ identity matrix in some way, e.g.,

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

are 4×4 permutation matrices. Note that if $x \in \mathbb{C}^m$ then Px permutes the elements of x in the same manner as the columns were permuted. If $A \in \mathbb{C}^{m \times m}$ then PA permutes the rows and AP permutes the columns of A.

(a) Determine the 4×4 permutation matrix for which

$$Px = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_1 \end{bmatrix}.$$

(b) Show that any permutation matrix is unitary.

(c) Determine $(P_1 P_2)^{-1}$ for the above P_1 and P_2.

Solution:

Problem #4.

Suppose $A \in \mathbb{C}^{m \times m}$ and $A^*A = D$ is a diagonal matrix with diagonal elements d_i, for $i = 1,\ 2,\ \ldots,\ m$. Determine D^{-1} and show that $A^{-1} = D^{-1}A^*$.

Solution:

Problem #5.

Let

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 0 & 2 & -4 \\ 1 & 2 & 2 \end{bmatrix}.$$

(a) Using Problem 4, determine A^{-1}.

(b) Define the polynomials

$$p_1(x) = x,$$
$$p_2(x) = 2,$$
$$p_3(x) = 6x^2 - 4.$$

Determine the coefficients $c_1,\ c_2,\ c_3$ so that the polynomial

$$p(x) = c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x)$$

interpolates the data

$$p(-1) = 2, \quad p(0) = 3, \quad p(1) = 0.$$

Solution:

Problem #6.

Let $u = \begin{bmatrix} 1+i \\ 2-i \end{bmatrix}$, where $i = \sqrt{-1}$.

(a) Compute u^*u and $\|u\|$.

(b) Compute uu^* and confirm that $(uu^*)^* = uu^*$.

(c) Determine a basis for range(uu^*).

(d) Determine a basis for null(uu^*).

Solution:

Problem #7.

Do Exercise 1.4 in the book.

Let $f_1,\ \ldots,\ f_8$ be a set of functions defined on the interval $[1, 8]$ with the property that for any numbers $d_1,\ \ldots,\ d_8$ there exists a set of coefficients $c_1,\ \ldots,\ c_8$ such that

$$\sum_{j=1}^{8} c_j f_j(i) = d_i, \qquad i = 1,\ \ldots,\ 8.$$

(a) Show by appealing to the theorems of this lecture that $d_1,\ \ldots,\ d_8$ determine $c_1,\ \ldots,\ c_8$ uniquely.

(b) Let A be the 8×8 matrix representing the linear mapping from data $d_1,\ \ldots,\ d_8$ to coefficients $c_1,\ \ldots,\ c_8$. What is the i, j entry of A^{-1}?

Solution:

Problem #8.

Do Exercise 3.1 in the book.

Prove that if W is an arbitrary nonsingular matrix, the function $\|\cdot\|_W$ defined by

$$\|x\|_W = \|Wx\|$$

is a vector norm (where $\|\cdot\|$ is an arbitrary vector norm).

Solution:

Problem #9.

Do Exercise 3.3 in the book. Hint: (c) and (d) follow from (a) and (b).

Vector and matrix p-norms are related by various inequalities, often involving the dimensions m or n. For each of the following, verify the inequality and give an example of a nonzero vector or matrix (for general m, n) for which equality is achieved. In this problem x is an m-vector and A is an $m \times n$ matrix.

(a) $\|x\|_\infty \leq \|x\|_2$,

(b) $\|x\|_2 \leq \sqrt{m}\|x\|_\infty$,

(c) $\|A\|_\infty \leq \sqrt{n}\|A\|_2$,

(d) $\|A\|_2 \leq \sqrt{m}\|A\|_\infty$.

Solution: