Each part of each problem is worth 8 points (96 points total).
Please show all of your work and justify all your answers.

1. (a) Show that for any $x \in \mathbb{C}^m$, $\|x\|_\infty \leq \|x\|_1$.

$$\|x\|_\infty = \max_i |x_i| \leq \sum_i |x_i| = \|x\|_1.$$

(b) Show that for any $x \in \mathbb{C}^m$, $\|x\|_1 \leq m\|x\|_\infty$.
If x_j is the element of x with maximum modulus then

$$\|x\|_1 = \sum_i |x_i| \leq m|x_j| = m\|x\|_\infty.$$

(c) Show that bounds of the form

$$c_1\|A\|_1 \leq \|A\|_\infty \leq c_2\|A\|_1$$

hold for any matrix $A \in \mathbb{C}^{m\times n}$, where the constants c_1 and c_2 depend only on m and n (not on the particular matrix). Determine these constants (the best possible, using the bounds from parts (a) and (b)). Hint: Tackle each inequality separately.

$$\|A\|_1 = \max_{x\neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \frac{\|Ay\|_1}{\|y\|_1} \leq \frac{m\|Ay\|_\infty}{\|y\|_\infty} \leq m\|A\|_\infty$$

where y is the vector that maximizes the ratio. So $c_1 = 1/m$.

$$\|A\|_\infty = \max_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \frac{\|Ay\|_\infty}{\|y\|_\infty} \leq \frac{n\|Ay\|_1}{\|y\|_1} \leq n\|A\|_1$$

where $y \in \mathbb{C}^n$ is the vector that maximizes the ratio. So $c_2 = n$. This uses

$$\|y\|_1 \leq n\|y\|_\infty \quad \Longrightarrow \quad \frac{1}{\|y\|_\infty} \leq \frac{n}{\|y\|_1}$$

since $y \in \mathbb{C}^n$.

2. Let

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -3 \\ 1 & 3 \\ 1 & -3 \end{bmatrix}.$$

(a) Determine the reduced QR factorization of the matrix A.
Normalizing the first column of A gives

$$q_1 = a_1/\|a_1\| = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Since $a_1^T a_2 = 0$ the columns are already orthogonal so we only need to normalize the second column to get q_2 by dividing by $\|a_2\| = 6$.
We find that

$$Q = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad R = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}.$$

(b) Using any method you wish, solve the least squares problem $Ax = b$ for

$$b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

You can form the normal equations $A^T Ax = A^T b$ or solve the system $Rx = Q^T b$. Either way you need only solve a *diagonal* system of two equations for

$$x = \begin{bmatrix} 3/4 \\ 1/12 \end{bmatrix}.$$

3. For this problem, let $P \in \mathbb{C}^{m\times m}$ be a nonzero projector, and let $Q \in \mathbb{C}^{m\times n}$ with $n < m$ and $Q^*Q = I_{n\times n}$.

(a) Is Q unitary? (Justify your answer.)
No, only a square matrix can be unitary. $Q^*Q = I$ but $QQ^* \neq I$.

(b) Show that $\|Q\|_2 = 1$.
For any x, $\|Qx\|^2 = x^*Q^*Qx = x^*x = \|x\|^2$ and hence

$$\frac{\|Qx\|}{\|x\|} = 1$$

for all x and so the norm is 1.

(c) Show that $\|P\|_2 \geq 1$.
Any projector satisfies $P^2 = P$. Choose x so that $y = Px \neq 0$. Then $P(Px) = P^2x = Px$ and hence $Py = y$ and $\|Py\|/\|y\| = 1$. The matrix norm is the maximum of this ratio and must be at least this large.
Note the problem stated that P is a *nonzero* projector. The zero matrix is a projector but has norm 0. The proof above would fail since there is no x for which $Px \neq 0$ in this case.

(d) Suppose P is an *orthogonal projector* (recall that this means $P = P^*$, not $P^*P = I$). Show that $\|P\|_2 = 1$. Hint: Write P in terms of a matrix with the properties of Q.
Any orthogonal projector can be written as $P = QQ^*$ where the columns of Q are an orthogonal basis for the range of P.
So $\|P\| \leq \|Q\|\,\|Q^*\|$. We know $\|Q\| \leq 1$ and also $\|A^*\| = \|A\|$ in the 2-norm for any A, so we also have $\|Q^*\| = 1$.

Note that in other norms it is not always true that $\|A^*\| = \|A\|$, since, for example $\|A^*\|_1 = \|A\|_\infty$. For the 2-norm you can verify that from the fact that

$$A = U\Sigma V^* \quad \Longrightarrow \quad A^* = V\Sigma U^*,$$

so both matrices have the same $\sigma_1 = \|A\|_2$.

For an orthogonal projector, you could also note that $P = QIQ^*$ is the SVD of P and so $\sigma_1 = 1$.

4. Let $A = e_1e_2^* + 2e_2e_3^* = e_1e_2^T + 2e_2e_3^T \in \mathbb{R}^{3\times 3}$, where $e_1,\ e_2$, and e_3 are the unit vectors in $\mathbb{R}^3$, i.e. the three columns of the 3×3 identity matrix.

   (a) What is the rank of A?

   The rank is 2. There are several ways to justify this. For example by noting that the singular values are $2,\ 1,\ 0$ and only 2 are nonzero.

   (b) Determine the full SVD of the matrix A. Hint: U and V will be permutation matrices. Another hint: What are the three singular values of A?

   Write $A = 2e_2e_3^* + e_1e_2^* + 0e_3e_1^*$ to see that

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

   Remember that the proper SVD form requires the σ_i in decreasing order.

   (c) Determine the projection matrix P that orthogonally projects any vector in $\mathbb{R}^3$ onto the range of A.

   $P = \hat{U}\hat{U}^*$ where $\hat{U}$ is the reduced U with only the columns corresponding to nonzero singular values. These columns form an orthogonal basis for the range of A.

$$\hat{U} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

   You could also deduce this from the original form of A given: Ax is a linear combination of e_1 and e_2 for any x, so P must project onto the x_1–x_2 plane.

   Note that if you use the full U instead of $\hat{U}$ you would get $P = I$, which can't be right since it has rank 3 instead of 2. Since U is unitary its columns span all of $\mathbb{R}^3$.