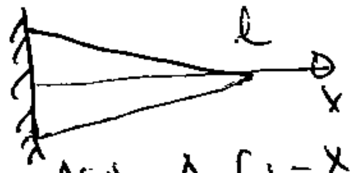
# Approximate Methods – Continuous Systems. 

**Ex 1** Longitudinal oscillations of a tapered rod (also see p. 531)

Obtain the two lowest natural freqs.



$A(x) = A_0\left(1 - \frac{x}{l}\right)$

Eqn of motion:

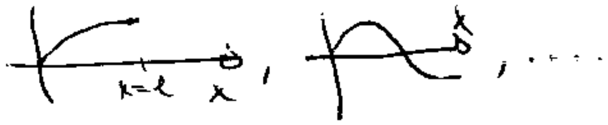
$$\frac{\partial}{\partial x}\left[A(x)E\frac{\partial u}{\partial x}\right] = \rho A(x)\frac{\partial^2 u}{\partial t^2}$$

$u$ is the displacement along $x$

so $L = -\frac{\partial}{\partial x}\left[A(x)E\frac{\partial}{\partial x}\right]$ and

$M = \rho A(x)$

Let $U^{(n)} = \sum_{i=1}^{n} a_i \sin\frac{(2i-1)}{2l}\pi x$ i.e.



so here $V_i = \sin\left(\frac{2i-1}{2l}\pi x\right)$ (or $\phi_i$ in the book)

$$k_{ij} = \int_0^l V_i L[V_j]\,dx = \int_0^l \sin\left(\frac{2i-1}{2l}\pi x\right)\frac{\partial}{\partial x}\left[A_0\left(1-\frac{x}{l}\right)E\frac{\partial}{\partial x}\left(\sin\frac{2j-1}{2l}\pi x\right)\right]dx$$

integ by parts

$$k_{ij} = EA_0\int_0^l \left(1-\frac{x}{l}\right)\left(\frac{2i-1}{2l}\pi\right)\cos\left[\frac{2i-1}{2l}\pi x\right]\frac{\partial}{\partial x}\left(\sin\left[\frac{2j-1}{2l}\pi x\right]\right)dx$$

$$k_{ij} = EA_0\frac{(2i-1)(2j-1)}{4l^2}\int_0^l\left(1-\frac{x}{l}\right)\cos\left(\frac{2i-1}{2l}\pi x\right)\cos\left(\frac{2j-1}{2l}\pi x\right)dx$$

if we use two terms in $U^{(n)}$

$$k_{11} = \frac{EA_0}{4l^2}\int_0^l \cos^2\frac{\pi x}{2l}\left(1-\frac{x}{l}\right)dx = \frac{EA_0}{4l}\left(1+\frac{\pi^2}{4}\right)$$

$$k_{22} = \frac{EA_0}{4l^2}\int_0^l \cos^2\frac{3\pi}{2l}\left(1-\frac{x}{l}\right)dx = \frac{EA_0}{4l}\left(4+9\pi^2\right),\quad k_{12}=k_{21}=\frac{3EA_0}{4l}$$

also:

$$m_{ij} = \int_0^l V_i M[V_j]\,dx = \rho A_0\int_0^l V_i V_j\left(1-\frac{x}{l}\right)dx =$$

$$= \rho A_0\int_0^l \sin\left(\frac{2i-1}{2l}\pi x\right)\sin\left(\frac{2j-1}{2l}\pi x\right)dX$$

Ex 1 cont)

$$m_{11} = \frac{\rho A_0 l}{4}\left(1 - \frac{4}{\pi^2}\right), \quad m_{22} = \frac{\rho A_0 l}{4}\left(1 - \frac{4}{9\pi^2}\right)$$

$$m_{12} = m_{21} = \frac{\rho A_0 l}{\pi^2}$$

From $\left( \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} - \lambda^{(2)} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \right) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{vmatrix} \frac{EA_0}{4l}\left(1+\frac{\pi^2}{4}\right) - \lambda^{(2)}\frac{\rho A_0 l}{4}\left(1-\frac{4}{\pi^2}\right), & \frac{3EA_0}{4l} - \lambda^{(2)}\frac{\rho A_0 l}{\pi^2} \\ \frac{3EA_0}{4l} - \lambda^{(2)}\frac{\rho A_0 l}{\pi^2}, & \frac{EA_0}{4l}\left(4+9\pi^2\right) - \lambda^{(2)}\frac{\rho A_0 l}{4}\left(1-\frac{4}{9\pi^2}\right) \end{vmatrix} = 0$$

solve quadratic $\Rightarrow$

$$\Rightarrow \omega_1^{(2)} = \sqrt{\lambda_1^{(2)}} = 2.4062 \frac{1}{l}\sqrt{\frac{E}{\rho}}$$

$$\omega_2^{(2)} = \sqrt{\lambda_2^{(2)}} = 5.5924 \frac{1}{l}\sqrt{\frac{E}{\rho}}$$

$\Leftarrow$ Two lowest nat. freqs